Regularization errors in the one-fluid formulation

Daniel Fuster

Y. Mimoh, E. Sultan
Institut ∂'Alembert (Paris)

July 7-9th, 2025
BGUM 2025 University of Oxford (UK)

Motivation

Singularities and jumps naturally appear in physical systems

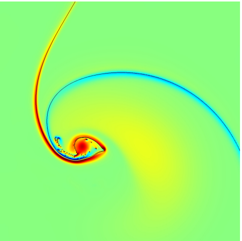
How to deal with them numerically??

Navier-Stokes equations

Sharp limit equations

$$\frac{\partial \boldsymbol{u}_i}{\partial t} + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i = -\frac{1}{\rho_i}\nabla p_i + \nu_i \nabla \cdot (2\boldsymbol{D}_i)$$

$$\nabla \cdot \left(\frac{1}{\rho_i}\nabla p_i\right) = \nabla \cdot (\boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i)$$



Jump conditions $[A] = A_2 - A_1$

$$[\boldsymbol{u}] = \dot{m}(1/\rho_1 - 1/\rho_2)$$

$$[\mu \boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{t}] = 0$$

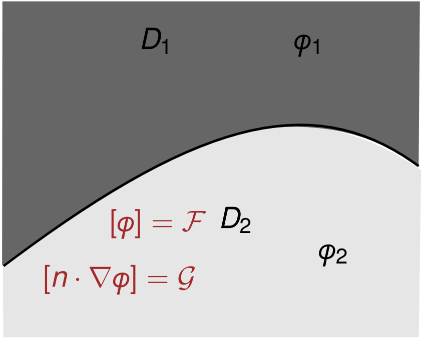
$$[p] = -\sigma\kappa + 2[\mu \boldsymbol{n}_l \boldsymbol{D} \boldsymbol{n}_l] + \ldots$$

$$[\frac{1}{\rho}\boldsymbol{n} \cdot \nabla p] = [\frac{1}{\rho}\boldsymbol{n} \cdot (\nabla \cdot (2\mu \boldsymbol{D}))]$$

Motivation

Sharp interface

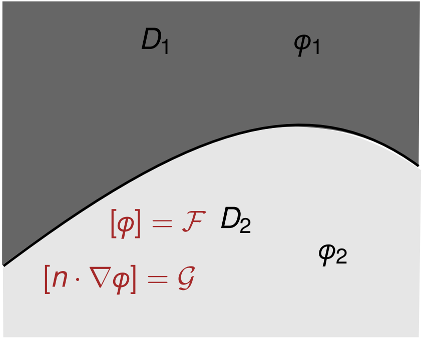
$\nabla \cdot (D_i \nabla \varphi_i) = s$



Motivation

Sharp interface

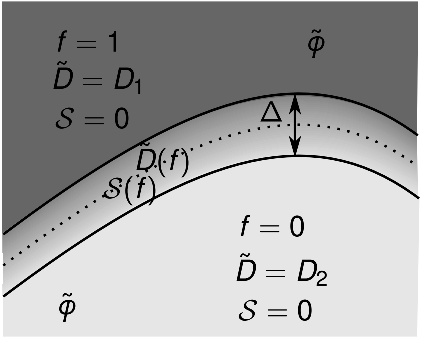
$$\nabla \cdot (D_i \nabla \varphi_i) = s$$



One fluid model

$$f = 0.5 - \frac{n}{\Delta}, \quad n \in [-\Delta/2; \Delta/2]$$

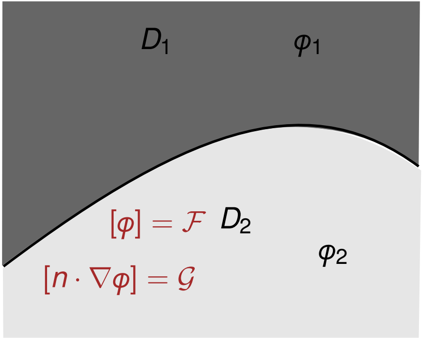
$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s + \mathcal{S}$$



Motivation

Sharp interface

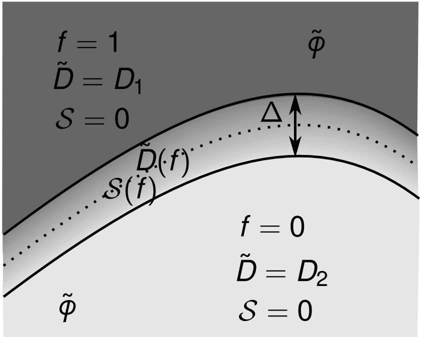
$$\nabla \cdot (D_i \nabla \varphi_i) = s$$



One fluid model

$f = 0.5 - \frac{n}{\Delta}, \quad n \in [-\Delta/2; \Delta/2]$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s + \mathcal{S}$$

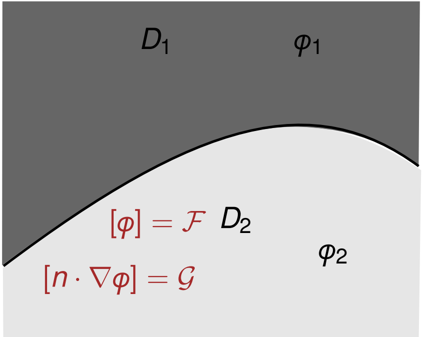


It filters φ with bandwidth Δ along discontinuities by:

Motivation

Sharp interface

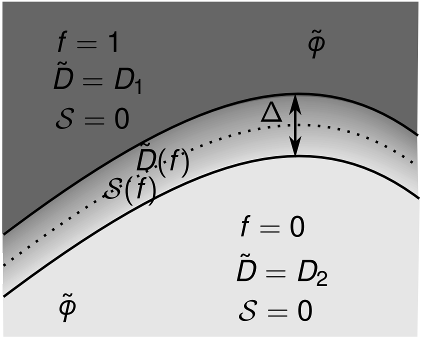
$\nabla \cdot (D_i \nabla \varphi_i) = s$



One fluid model

$f = 0.5 - \frac{n}{\Delta}, \quad n \in [-\Delta/2; \Delta/2]$

$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s + \mathcal{S}$



It filters φ with bandwidth Δ along discontinuities by:

Defining averaged properties: $\tilde{D}(f)$

Replacing jump conditions by aritificial sources $\mathcal{S}(f, \Delta, \mathcal{F}, \mathcal{G})$

Capturing jumps

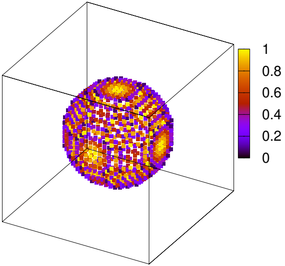
Consider the Poisson equation $\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$

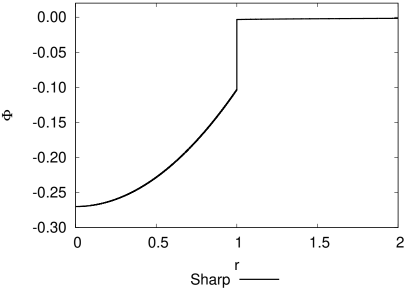
Vble jump: $[\varphi] = \mathcal{F} = 1$

Flux jump: $[\boldsymbol{n} \cdot \nabla \varphi] = 0$

Analytical solution:

s=1 inside, $D_1 \neq D_2$



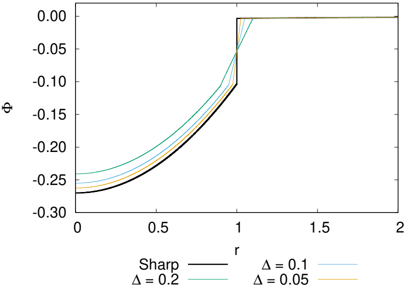


Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Analytical solution: Δ



Capturing jumps

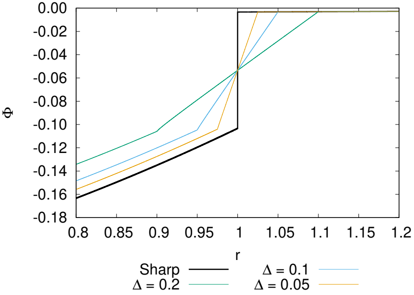
Inner region

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Analytical solution: Δ

ZOOMED VIEW $\rightarrow$



Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

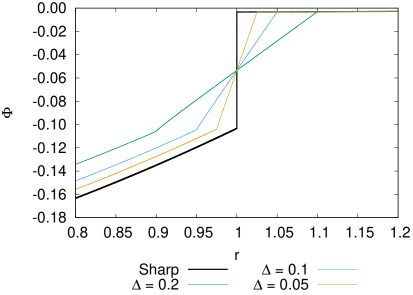
Inner region

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

- L_1 error norm $\sim \Delta$
- L_∞ norm does not converge
- Derivatives diverge

Analytical solution: Δ

ZOOMED VIEW $\rightarrow$



Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

Inner region

L_1 error norm $\sim \Delta$

L_∞ norm does not converge

Derivatives diverge

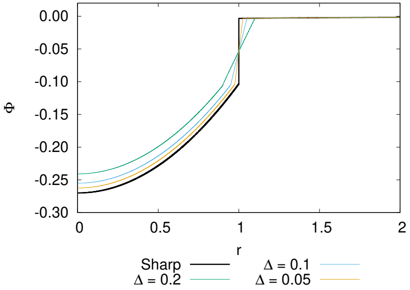
$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Outer region

Solution is $\mathcal{O}(\Delta)$

Variables have physical meaning

Analytical solution: Δ



Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Inner region

- L_1 error norm $\sim \Delta$
- L_∞ norm does not converge
- Derivatives diverge

Outer region

- Solution is $\mathcal{O}(\Delta)$
- Variables have physical meaning

Numerical solution

h: grid size

Δ: Reg length

Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

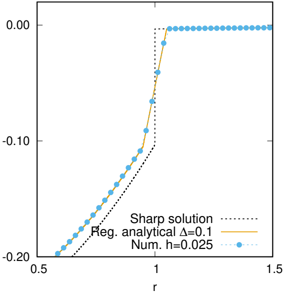
Inner region

- L_1 error norm $\sim \Delta$
- L_∞ norm does not converge
- Derivatives diverge

Outer region

- Solution is $\mathcal{O}(\Delta)$
- Variables have physical meaning

Numerical solution



Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

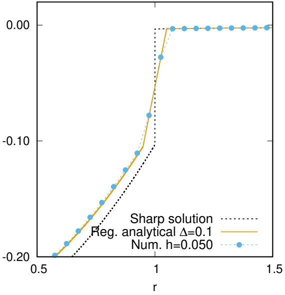
Inner region

- L_1 error norm $\sim \Delta$
- L_∞ norm does not converge
- Derivatives diverge

Outer region

- Solution is $\mathcal{O}(\Delta)$
- Variables have physical meaning

Numerical solution



Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

Inner region

L_1 error norm $\sim \Delta$

L_∞ norm does not converge

Derivatives diverge

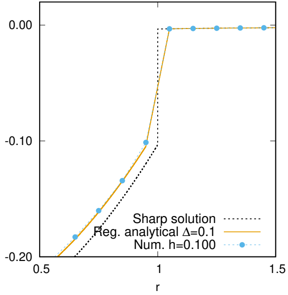
$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Outer region

Solution is $\mathcal{O}(\Delta)$

Variables have physical meaning

Numerical solution



Capturing jumps

$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Inner region

- L_1 error norm $\sim \Delta$
- L_∞ norm does not converge
- Derivatives diverge

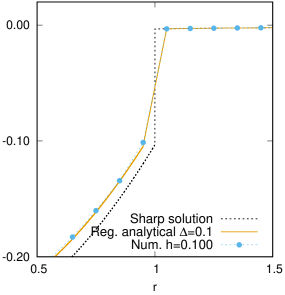
Outer region

- Solution is $\mathcal{O}(\Delta)$
- Variables have physical meaning

Regularization controls

even in the limit $h = \Delta$

Numerical solution



Capturing jumps

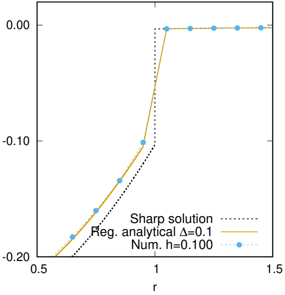
$$\nabla \cdot (D_i \nabla \varphi_i) = s_i \qquad i = 1, 2$$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = s_i + \nabla \cdot (-\mathcal{F} \tilde{D} \nabla f)$$

Inner region

- L_1 error norm $\sim \Delta$
- L_∞ norm does not converge
- Derivatives diverge

Numerical solution



Outer region

- Solution is $\mathcal{O}(\Delta)$
- Variables have physical meaning

Regularization controls

even in the limit $h = \Delta$

Can we do better?

Another analytical example

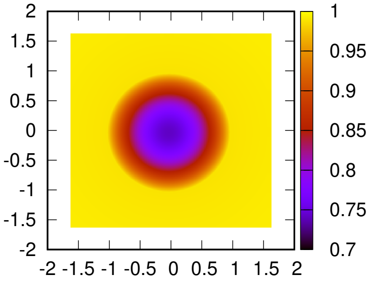
$D_1 \neq D_2$ is sufficient to have $\mathcal{O}(\Delta)$ errors

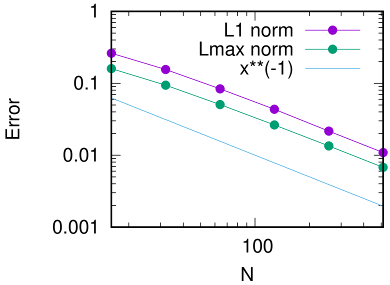
$$\nabla \cdot (D_1 \nabla \varphi_1) = 1 \qquad [\varphi] = 0$$

$$\nabla \cdot (D_2 \nabla \varphi_2) = 0 \qquad [D\frac{\partial \varphi}{\partial n}] = 0$$

$D_1 = 1$ $D_2 = 10$ Derivatives discontinuous

VOF (sharp?) + Second order solver= **1st order convergence**

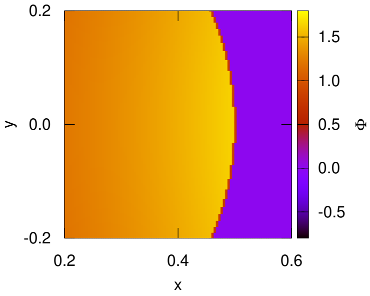


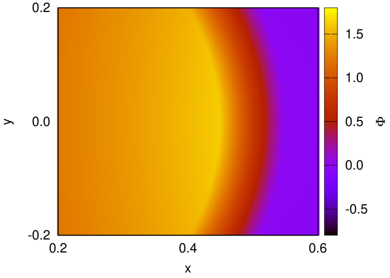


Problem setup

HOW DOES THE ERROR BEHAVE?

$$\epsilon_\delta = \varphi_i - \tilde{\varphi}$$





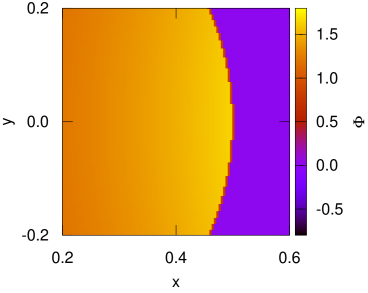
Problem setup

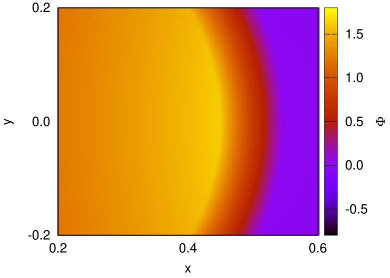
HOW DOES THE ERROR BEHAVE?

$$\epsilon_\delta = \varphi_i - \tilde{\varphi}$$

DOES ERROR STAY INSIDE?

IS FILTERING IRREVERSIBLE?





Three problems:

1) $\nabla \cdot (D_i \nabla \varphi_i) = s$ $\qquad$ $\epsilon = \varphi_i - \tilde{\varphi}$

Jump conditions at n=0

$[\varphi^{sharp}] = \mathcal{F}(\boldsymbol{x}_I)$

$[D \frac{\partial \varphi^{sharp}}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I)$

Three problems:

1) $\nabla \cdot (D_i \nabla \varphi_i) = s$ $\epsilon = \varphi_i - \tilde{\varphi}$

2) $\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = \tilde{s}$ $\tilde{s} = s + \mathcal{S}$ (arbitrary $\mathcal{S}$ artificial model)

Jump conditions at n=0

$[\varphi^{sharp}] = \mathcal{F}(\boldsymbol{x}_I)$ $[\tilde{\varphi}] = 0$

$[D\frac{\partial \varphi^{sharp}}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I)$ $\tilde{D}[\frac{\partial \tilde{\varphi}}{\partial n}] = 0$

Three problems:

1) $\nabla \cdot (D_i \nabla \varphi_i) = s$ $\qquad \epsilon = \varphi_i - \tilde{\varphi}$

2) **–** $\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = \tilde{s}$ $\qquad \tilde{s} = s + \mathcal{S}$ (arbitrary $\mathcal{S}$ artificial model)

---

3) $\boxed{\nabla \cdot (D_i \nabla \epsilon_i) = \nabla \cdot ((D_i - \tilde{D}) \nabla \tilde{\varphi}) - (\tilde{s} - s_i)}$

Jump conditions at n=0

$[\varphi^{sharp}] = \mathcal{F}(\boldsymbol{x}_I)$ $\qquad [\tilde{\varphi}] = 0$ $\qquad [\epsilon] = \mathcal{F}(\boldsymbol{x}_I)$

$[D\frac{\partial \varphi^{sharp}}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I)$ $\qquad \tilde{D}[\frac{\partial \tilde{\varphi}}{\partial n}] = 0$ $\qquad [D\frac{\partial \epsilon}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I) - [D]\frac{\partial \tilde{\varphi}}{\partial n}$

Three problems:

1) $\nabla \cdot (D_i \nabla \varphi_i) = s$ $\epsilon = \varphi_i - \tilde{\varphi}$

2) – $\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = \tilde{s}$ $\tilde{s} = s + \mathcal{S}$ (arbitrary $\mathcal{S}$ artificial model)

---

3) $$\nabla \cdot (D_i \nabla \epsilon_i) = \nabla \cdot ((D_i - \tilde{D}) \nabla \tilde{\varphi}) - (\tilde{s} - s_i)$$

Jump conditions at n=0

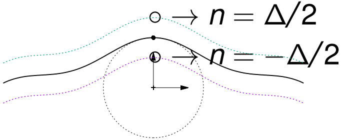
$[\varphi^{sharp}] = \mathcal{F}(\boldsymbol{x}_I)$ $[\tilde{\varphi}] = 0$ $[\epsilon] = \mathcal{F}(\boldsymbol{x}_I)$

$[D \frac{\partial \varphi^{sharp}}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I)$ $\tilde{D}[\frac{\partial \tilde{\varphi}}{\partial n}] = 0$ $[D \frac{\partial \epsilon}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I) - [D] \frac{\partial \tilde{\varphi}}{\partial n}$

:-) The inverse problem is closed!

...How to solve for it???

Error equation



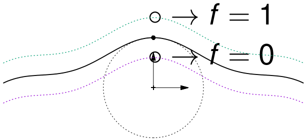
Inner Problem

$$\nabla \cdot (D_i \nabla \epsilon_i) = \nabla \cdot ((D_i - \tilde{D}) \nabla \tilde{\varphi}) - (\tilde{s} - s_i)$$

$$[\epsilon] = \mathcal{F}(\boldsymbol{x}_I)$$

$$[D \tfrac{\partial \epsilon}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I) - [D] \tfrac{\partial \tilde{\varphi}}{\partial n}$$

Error equation



<u>Inner Problem</u> $\boldsymbol{x} \rightarrow (\boldsymbol{x}_I, f)$

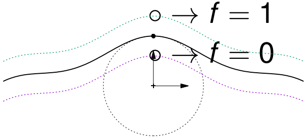
$$\nabla \cdot (D_i \nabla \epsilon_i) = \nabla \cdot ((D_i - \tilde{D}) \nabla \tilde{\varphi}) - (\tilde{s} - s_i)$$

$$[\epsilon] = \mathcal{F}(\boldsymbol{x}_I)$$

$$[D \tfrac{\partial \epsilon}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I) - [D] \tfrac{\partial \tilde{\varphi}}{\partial n}$$

Error equation

$\nabla \cdot (D_1 \nabla \epsilon_1) = 0$ Outer Problem fluid 1



Inner Problem $\boldsymbol{x} \rightarrow (\boldsymbol{x}_I, f)$

$$\nabla \cdot (D_i \nabla \epsilon_i) = \nabla \cdot ((D_i - \tilde{D}) \nabla \tilde{\varphi}) - (\tilde{s} - s_i)$$

$$[\epsilon] = \mathcal{F}(\boldsymbol{x}_I)$$

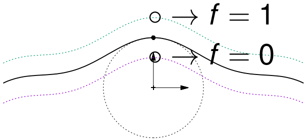
$$[D \frac{\partial \epsilon}{\partial n}] = \mathcal{G}(\boldsymbol{x}_I) - [D] \frac{\partial \tilde{\varphi}}{\partial n}$$

$\nabla \cdot (D_2 \nabla \epsilon_2) = 0$ Outer Problem fluid 2

Error equation

To solve for the outer problem we just need effective jump conditions

$\nabla \cdot (D_1 \nabla \epsilon_1) = 0$ Outer Problem fluid 1



$$[[\epsilon']] = \tilde{\epsilon}(f = 0) - \tilde{\epsilon}(f = 1)$$

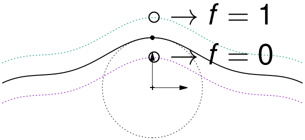
$$[[D \frac{\partial \epsilon'}{\partial n}]] = D_2 \frac{\partial \tilde{\epsilon}}{\partial n}\bigg|_{f=0} - D_2 \frac{\partial \tilde{\epsilon}}{\partial n}\bigg|_{f=1}$$

$\nabla \cdot (D_2 \nabla \epsilon_2) = 0$ Outer Problem fluid 2

Error equation

To solve for the outer problem we just need effective jump conditions

$$\nabla \cdot (D_1 \nabla \epsilon_1) = 0$$ Outer Problem fluid 1



$$[[\epsilon']] = \tilde{\epsilon}(f = 0) - \tilde{\epsilon}(f = 1)$$

$$[[D \frac{\partial \epsilon'}{\partial n}]] = D_2 \frac{\partial \tilde{\epsilon}}{\partial n}\Big|_{f=0} - D_2 \frac{\partial \tilde{\epsilon}}{\partial n}\Big|_{f=1}$$

$$\nabla \cdot (D_2 \nabla \epsilon_2) = 0$$ Outer Problem fluid 2

We integrate the error equation along the normal direction to get jumps

The outer solution fixes the integration constant

Expansion of the Regularized solution in the inner region

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = \tilde{s} \rightarrow$$

$$\tilde{J}_n \equiv -\frac{1}{\Delta} \tilde{D} \frac{\partial \tilde{\varphi}}{\partial f}$$

$$\frac{\partial \tilde{J}_n}{\partial f} - \mathcal{K} \Delta \tilde{J}_n = \left(\tilde{D} \nabla_s \tilde{\varphi} - s_i - \mathcal{S} \right) \Delta$$

$\tilde{D}(f)$ and $\mathcal{S}$ determines the structure of $\tilde{\varphi}$ in the inner region

and ultimately the structure of the error fields associated

$$\tilde{\varphi} \rightarrow \tilde{\epsilon}_i = \varphi_i - \tilde{\varphi}$$

$$\tilde{J}_n \rightarrow \frac{\mathcal{J}}{D_i} = \frac{\partial \varphi}{\partial n} - \frac{\partial \tilde{\varphi}}{\partial n}$$

Expansion of the Regularized solution in the inner region

$$\nabla \cdot (\tilde{D}\nabla\tilde{\varphi}) = \tilde{s} \rightarrow$$

$$\tilde{J}_n \equiv -\frac{1}{\Delta}\tilde{D}\frac{\partial\tilde{\varphi}}{\partial f}$$

$$\frac{\partial \tilde{J}_n}{\partial f} - \mathcal{K}\Delta\tilde{J}_n = \left(\tilde{D}\nabla_s\tilde{\varphi} - s_i - \mathcal{S}\right)\Delta$$

$\tilde{D}(f)$ and $\mathcal{S}$ determines the structure of $\tilde{\varphi}$ in the inner region

and ultimately the structure of the error fields associated

Example

$$\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$$

$$\tilde{\varphi}(\boldsymbol{x}, \Delta) = \tilde{\varphi}^{(0)}(\boldsymbol{x}, f) + \tilde{\varphi}^{(1)}(\boldsymbol{x}, f)\Delta + \tilde{\varphi}^{(2)}(\boldsymbol{x}, f)\Delta^2 + \ldots.,$$

$$\tilde{J}_n(\boldsymbol{x}, \Delta) = \frac{\tilde{J}_n^{(-1)}(\boldsymbol{x}, f)}{\Delta} + \tilde{J}_n^{(0)}(\boldsymbol{x}, f) + \tilde{J}^{(1)}(\boldsymbol{x}, f)\Delta + \tilde{J}_n^{(2)}(\boldsymbol{x}, f)\Delta^2 + \ldots.$$

$\mathcal{O}(\Delta^{-1})$ and $\mathcal{O}(1)$ solutions

Approx. $\mathcal{O}(1)$

$$\nabla \cdot (\tilde{D}\nabla\tilde{\varphi}) = \tilde{s}$$

Model $\tilde{s} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$

$$[\varphi] = \mathcal{F}$$

$$[\nabla\varphi \cdot \boldsymbol{n}_I] = \mathcal{G}$$

Outer error problem

$$[[\epsilon_i']] = \mathcal{O}(\Delta)$$

$$[[\mathcal{J}_{\epsilon_i}]] = \mathcal{O}(\Delta)$$

Approx. $\mathcal{O}(1)$

$$\nabla \cdot (\tilde{D} \nabla \tilde{\varphi}) = \tilde{s}$$

Model $\tilde{s} = -\nabla \cdot (\tilde{D} \mathcal{F} \nabla f) + \mathcal{G} \delta_s$

$$[\varphi] = \mathcal{F}$$

$$[\nabla \varphi \cdot \boldsymbol{n}_I] = \mathcal{G}$$

Outer error problem

$$[[\epsilon_i']] = \mathcal{O}(\Delta)$$

$$[[\mathcal{J}_{\epsilon_i}]] = \mathcal{O}(\Delta)$$

Inner error problem

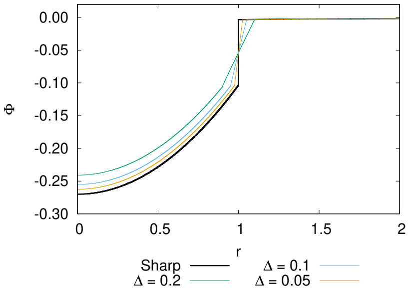
$$\tilde{\epsilon} \approx \mathcal{F}(f - f_i) + \mathcal{O}(\Delta)$$

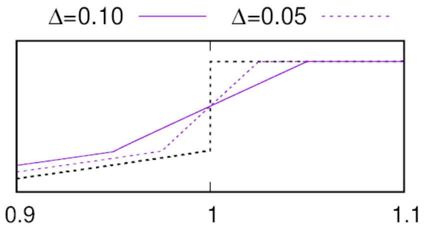
$$\frac{\partial \varphi_i}{\partial n} - \frac{\partial \tilde{\varphi}}{\partial n} \approx -\frac{1}{\Delta} D_i \mathcal{F} \qquad + \left(\frac{1}{D_i} - \frac{1}{\tilde{D}} \right) \tilde{J}_n^{(0)} + \mathcal{G}(f - f_i) + \mathcal{O}(\Delta)$$

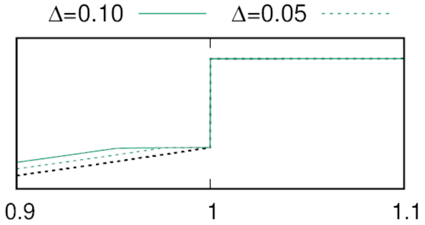
Contributions due to $[\varphi]$, $[\nabla \varphi \cdot \boldsymbol{n}_I]$ and $[D]$

$\mathcal{O}(\Delta)$ inner solution reconstruction for primitive variables:

$$\boxed{\varphi_i \approx \tilde{\varphi} + \mathcal{F}(f - f_i)} + \mathcal{O}(\Delta)$$

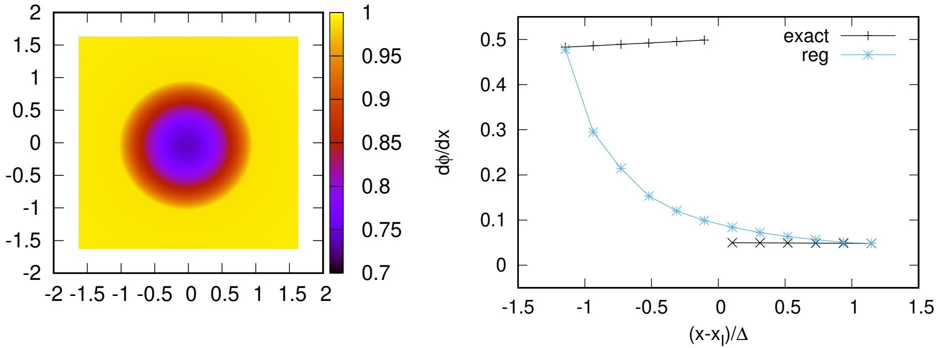






Example of $\mathcal{O}(\Delta)$ inner solution reconstruction **<u>for derivatives</u>**

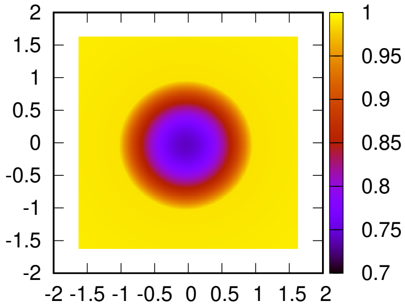
$$\frac{\partial \varphi_i}{\partial n} \approx \frac{\partial \tilde{\varphi}}{\partial n} + \mathcal{O}(1)$$

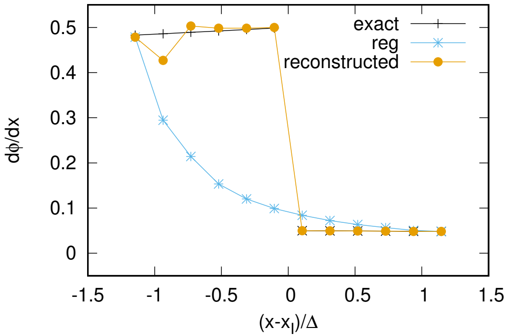


NON PHYSICAL!

Example of $\mathcal{O}(\Delta)$ inner solution reconstruction **for derivatives**

$$\frac{\partial \varphi_i}{\partial n} \approx \frac{\partial \tilde{\varphi}}{\partial n} - \frac{\mathcal{F}}{\Delta} + \left(\frac{1}{D_i} - \frac{1}{\tilde{D}}\right)\left(\tilde{J}_n - \frac{\mathcal{F}\tilde{D}}{\Delta}\right) + \frac{\mathcal{G}}{D_i}(f - f_i) + \mathcal{O}(\Delta)$$





PHYSICAL!

$\mathcal{O}(\Delta)$ solution

Approx. $\mathcal{O}(\Delta)$

Model generalization $\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$

Outer error correction $\nabla \cdot (D_i \nabla \epsilon_i') = 0$

$$[[\epsilon'^{(1)}]] = \tilde{J}_n(\boldsymbol{x}_I) f_1(\tilde{D}) - \frac{\mathcal{F}(\boldsymbol{x}_I)}{\Delta} \tilde{D} f_1(\tilde{D}) + \mathcal{G}(\boldsymbol{x}_I) f_2(\tilde{D})$$

$$[[\mathcal{J}_\epsilon'^{(1)}]] = \mathcal{L}^t(\tilde{\varphi})\Big|_{n=0} f_3(\tilde{D}) - \mathcal{L}^t(\mathcal{F}) f_4(\tilde{D})$$

$f_i(\tilde{D})$: are functions of the regularization law

Approx. $\mathcal{O}(\Delta)$

Model generalization $\quad \mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$

Outer error correction $\nabla \cdot (D_i \nabla \epsilon_i') = 0$

$$[[\epsilon'^{(1)}]] = \tilde{J}_n(\boldsymbol{x}_I) f_1(\tilde{D}) - \frac{\mathcal{F}(\boldsymbol{x}_I)}{\Delta}\tilde{D} f_1(\tilde{D}) + \mathcal{G}(\boldsymbol{x}_I) f_2(\tilde{D})$$

$$[[\mathcal{J}_\epsilon'^{(1)}]] = \mathcal{L}^t(\tilde{\varphi})\Big|_{n=0} f_3(\tilde{D}) - \mathcal{L}^t(\mathcal{F}) f_4(\tilde{D})$$

$f_i(\tilde{D})$: are functions of the regularization law

Error *escapes* into the bulk at $\mathcal{O}(\Delta)$

Approx. $\mathcal{O}(\Delta)$

Model generalization $\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$

Outer error correction $\nabla \cdot (D_i \nabla \epsilon_i') = 0$

$$[[\epsilon'^{(1)}]] = \tilde{J}_n(\boldsymbol{x}_I) f_1(\tilde{D}) - \frac{\mathcal{F}(\boldsymbol{x}_I)}{\Delta}\tilde{D} f_1(\tilde{D}) + \mathcal{G}(\boldsymbol{x}_I) f_2(\tilde{D})$$

$$[[\mathcal{J}_\epsilon'^{(1)}]] = \mathcal{L}^t(\tilde{\varphi})\Big|_{n=0} f_3(\tilde{D}) - \mathcal{L}^t(\mathcal{F}) f_4(\tilde{D})$$

$f_i(\tilde{D})$: are functions of the regularization law

Error *escapes* into the bulk at $\mathcal{O}(\Delta)$ Need to solve a PDE to obtain it

Approx. $\mathcal{O}(\Delta)$

Model generalization $\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$

Outer error correction $\nabla \cdot (D_i \nabla \epsilon_i') = 0$

$$[[\epsilon'^{(1)}]] = \tilde{J}_n(\boldsymbol{x}_I) f_1(\tilde{D}) - \frac{\mathcal{F}(\boldsymbol{x}_I)}{\Delta} \tilde{D} f_1(\tilde{D}) + \mathcal{G}(\boldsymbol{x}_I) f_2(\tilde{D})$$

$$[[\mathcal{J}_\epsilon'^{(1)}]] = \mathcal{L}^t(\tilde{\varphi})\Big|_{n=0} f_3(\tilde{D}) - \mathcal{L}^t(\mathcal{F}) f_4(\tilde{D})$$

$f_i(\tilde{D})$: are functions of the regularization law

Error *escapes* into the bulk at $\mathcal{O}(\Delta)$ Need to solve a PDE to obtain it

The error depends on the solution/jump structure

but not on **curvatuve!!**

Approx. $\mathcal{O}(\Delta)$

Model generalization $\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}\delta_s$

Outer error correction $\nabla \cdot (D_i \nabla \epsilon_i') = 0$

$$[[\epsilon'^{(1)}]] = \tilde{J}_n(\boldsymbol{x}_I) f_1(\tilde{D}) - \frac{\mathcal{F}(\boldsymbol{x}_I)}{\Delta} \tilde{D} f_1(\tilde{D}) + \mathcal{G}(\boldsymbol{x}_I) f_2(\tilde{D})$$

$$[[\mathcal{J}_\epsilon'^{(1)}]] = \mathcal{L}^t(\tilde{\varphi})\Big|_{n=0} f_3(\tilde{D}) - \mathcal{L}^t(\mathcal{F}) f_4(\tilde{D})$$

$f_i(\tilde{D})$: are functions of the regularization law

Error *escapes* into the bulk at $\mathcal{O}(\Delta)$ Need to solve a PDE to obtain it

The error depends on the solution/jump structure

but not on **curvatuve!!**

Not general/optimal averaging rule exists

Approx. $\mathcal{O}(\Delta)$

Model generalization $\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}(\boldsymbol{x}_I)\delta_s$

Inner error correction

$$\tilde{\epsilon}_i^{(1)} = \epsilon_i'(n_i) - \Delta \int_{f_i}^{f} \left(\frac{1}{D_i} - \frac{1}{\tilde{D}}\right) \tilde{J}_n^{(0)} df - \Delta\mathcal{G}(\boldsymbol{x}_I)\frac{(f-f_i)^2}{2D_i}$$

$$\frac{\tilde{\mathcal{J}}_{\epsilon_i}^{(1)}}{D_i} = \frac{\mathcal{J}_{\epsilon_i}'(n_i)}{D_i} + \Delta\mathcal{L}^t(\mathcal{F}(\boldsymbol{x}_I))\frac{1}{2}(f - f_i)^2$$

Integration constants ϵ_i' $\mathcal{J}_{\epsilon_i}'$ depend on the outer problem

Depends on the regularization law

Approx. $\mathcal{O}(\Delta)$

Model generalization $\mathcal{S} = -\nabla \cdot (\tilde{D}\mathcal{F}\nabla f) + \mathcal{G}(\boldsymbol{x}_I)\delta_s$

Inner error correction

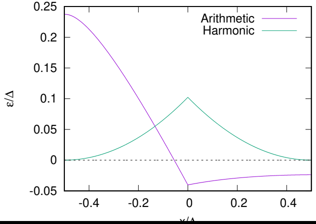
$$\tilde{\epsilon}_i^{(1)} = \epsilon_i'(n_i) - \Delta \int_{f_i}^{f} \left(\frac{1}{D_i} - \frac{1}{\tilde{D}}\right) \tilde{J}_n^{(0)} df - \Delta \mathcal{G}(\boldsymbol{x}_I)\frac{(f-f_i)^2}{2D_i}$$

$$\frac{\tilde{\mathcal{J}}_{\epsilon_i}^{(1)}}{D_i} = \frac{\mathcal{J}_{\epsilon_i}'(n_i)}{D_i} + \Delta \mathcal{L}^t(\mathcal{F}(\boldsymbol{x}_I))\frac{1}{2}(f - f_i)^2$$

Integration constants ϵ_i' $\mathcal{J}_{\epsilon_i}'$ depend on the outer problem

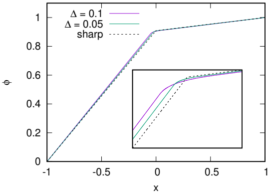
Depends on the regularization law

Example for $\mathcal{F} = 0$ and $\mathcal{G} = 0$



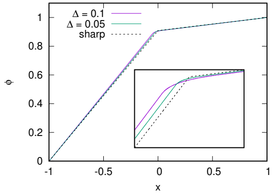
$\mathcal{O}(\Delta)$ correction ($\mathcal{O}(\Delta^2)$ reconstruction)

1) Compute solution

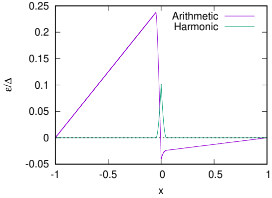


$\mathcal{O}(\Delta)$ correction ($\mathcal{O}(\Delta^2)$ reconstruction)

1) Compute solution

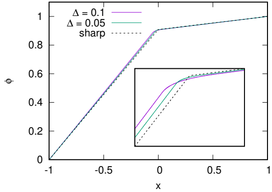


2)Solve for outer error problem

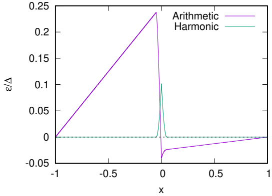


$\mathcal{O}(\Delta)$ correction ($\mathcal{O}(\Delta^2)$ reconstruction)

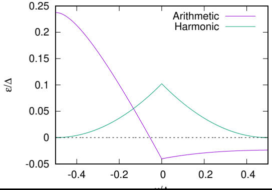
1) Compute solution



2)Solve for outer error problem



3) Solve for inner error problem



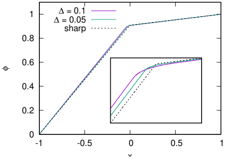
$\mathcal{O}(\Delta^n)$ solution

The same procedure can be generalized to arbitrary order to obtain the errors at order $\mathcal{O}(\Delta^n)$ from $\mathcal{O}(\Delta^{n-1})$

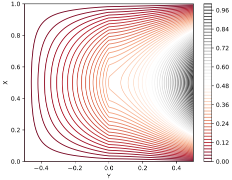
Analytical Validation (without jumps):

We analyze problems with interfaces where $\varphi^{(sharp)}$ and $\tilde{\varphi}$ can be analytically computed

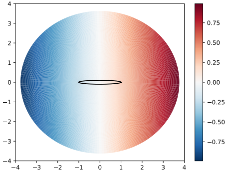
- 1D Laplace equation



- 1D Poisson equation



- 2D Laplace equation for plannar interface
- 2D Laplace equation for curved interface



[Fuster & Mimoh, JCP, 2024]

Analytical Validation (with jumps):

We analyze problems with interfaces where $\varphi^{(sharp)}$ and $\tilde{\varphi}$ can be analytically computed

- Sphere with variable jump

- Sphere with flux jump

- 2D Laplace equation with jump

- Complex problem

[Fuster & Sultan, under review]

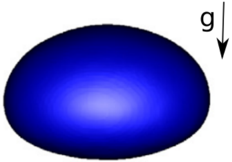
Applications:

To understand the variables influencing regularization errors

Applications:

To understand the variables influencing regularization errors

- Inside regularization variables (derivatives) can be unphysical
- First order errors are proportional to normal flux and surface Laplacian
- Errors related to curvature appear at second order!
- For $\kappa\Delta \ll 1$ a new AMR criterion is required??



Applications:

To understand the variables influencing regularization errors

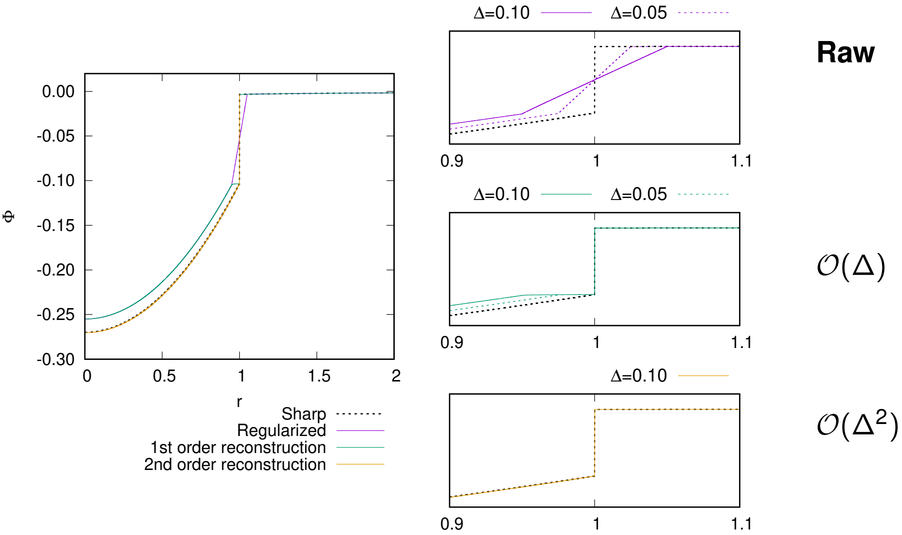
Inside regularization variables (derivatives) can be unphysical

First order errors are proportional to normal flux and surface Laplacian

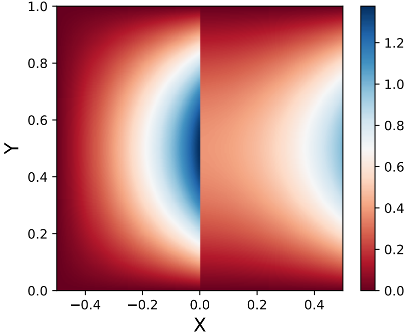
Errors related to curvature appear at second order!

To compensate for these errors to improve accuracy

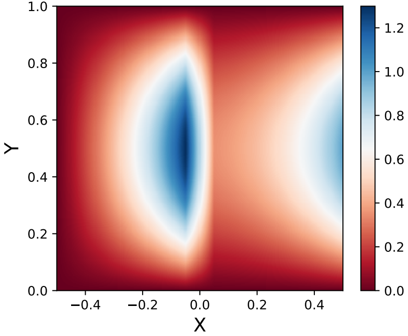
Example of inner solution reconstruction:



Laplace 2D with jump



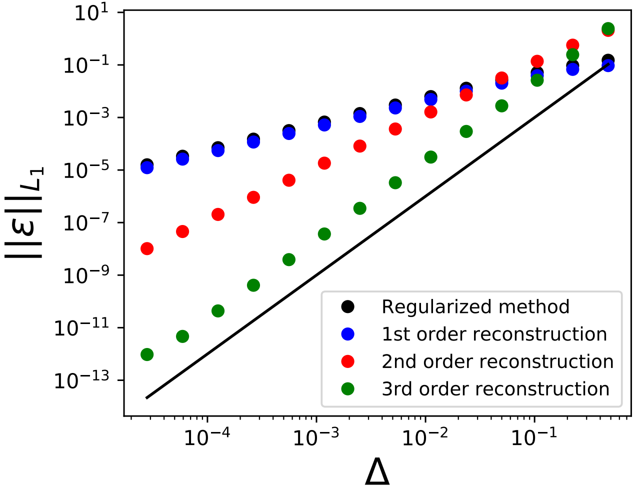
Original



Regularized

$\Delta = 0.1$

Laplace 2D with jump



Applications:

To understand the variables influencing regularization errors

- Inside regularization variables (derivatives) can be unphysical
- First order errors are proportional to normal flux and surface Laplacia
- Errors related to curvature appear at second order!

To compensate for these errors to improve accuracy

Correcting errors allow natural coupling between physical models

Applications:

To understand the variables influencing regularization errors

Inside regularization variables (derivatives) can be unphysical

First order errors are proportional to normal flux and surface Laplacia

Errors related to curvature appear at second order!

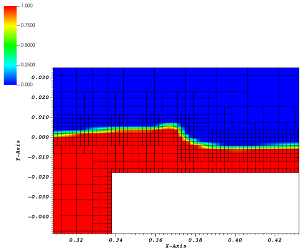
To compensate for these errors to improve accuracy

Correcting errors allow natural coupling between physical models

We can discuss the accuracy of different filtering techniques

and design optimal models for $\mathcal{S}$

Navier–Stokes



Artificial sources in the One fluid model

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\tilde{\rho}} \nabla p + \frac{1}{\tilde{\rho}} \nabla \cdot (2\tilde{\mu} \boldsymbol{D}) + \frac{\sigma \kappa}{\tilde{\rho}} \nabla f$$

$$\nabla \cdot \left(\frac{1}{\tilde{\rho}} \nabla p \right) = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla \cdot \left(\frac{1}{\tilde{\rho}} \nabla \cdot (2\tilde{\mu} \boldsymbol{D}) \right) + \nabla \cdot \left(\frac{\sigma \kappa}{\tilde{\rho}} \nabla f \right)$$

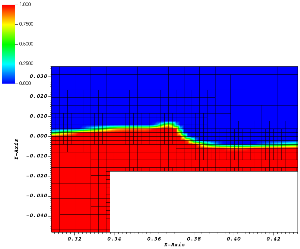
Jump conditions $[A] = A_2 - A_1$

$$[\boldsymbol{u}] = 0 \qquad [\mu \boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{t}] = 0$$

$$[p] = 0 \qquad [\frac{1}{\rho} \boldsymbol{n} \cdot \nabla p] = 0$$

Navier–Stokes

Artificial sources in the One fluid model



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\tilde{\rho}} \nabla p + \frac{1}{\tilde{\rho}} \nabla \cdot (2\tilde{\mu}\boldsymbol{D}) + \frac{\sigma\kappa}{\tilde{\rho}} \nabla f$$

$$\nabla \cdot \left(\frac{1}{\tilde{\rho}} \nabla p \right) = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla \cdot \left(\frac{1}{\tilde{\rho}} \nabla \cdot (2\tilde{\mu}\boldsymbol{D})\right) + \nabla \cdot \left(\frac{\sigma\kappa}{\tilde{\rho}} \nabla f\right)$$

Jump conditions $[A] = A_2 - A_1$

$[\boldsymbol{u}] = 0$ $\qquad$ $[\mu \boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{t}] = 0$

$[p] = 0$ $\qquad$ $[\frac{1}{\rho} \boldsymbol{n} \cdot \nabla p] = 0$

Accuracy of these models in the linear regime

Aknine's presentation (next!)