Towards self-similar solvers:

An application to surface tension driven flows

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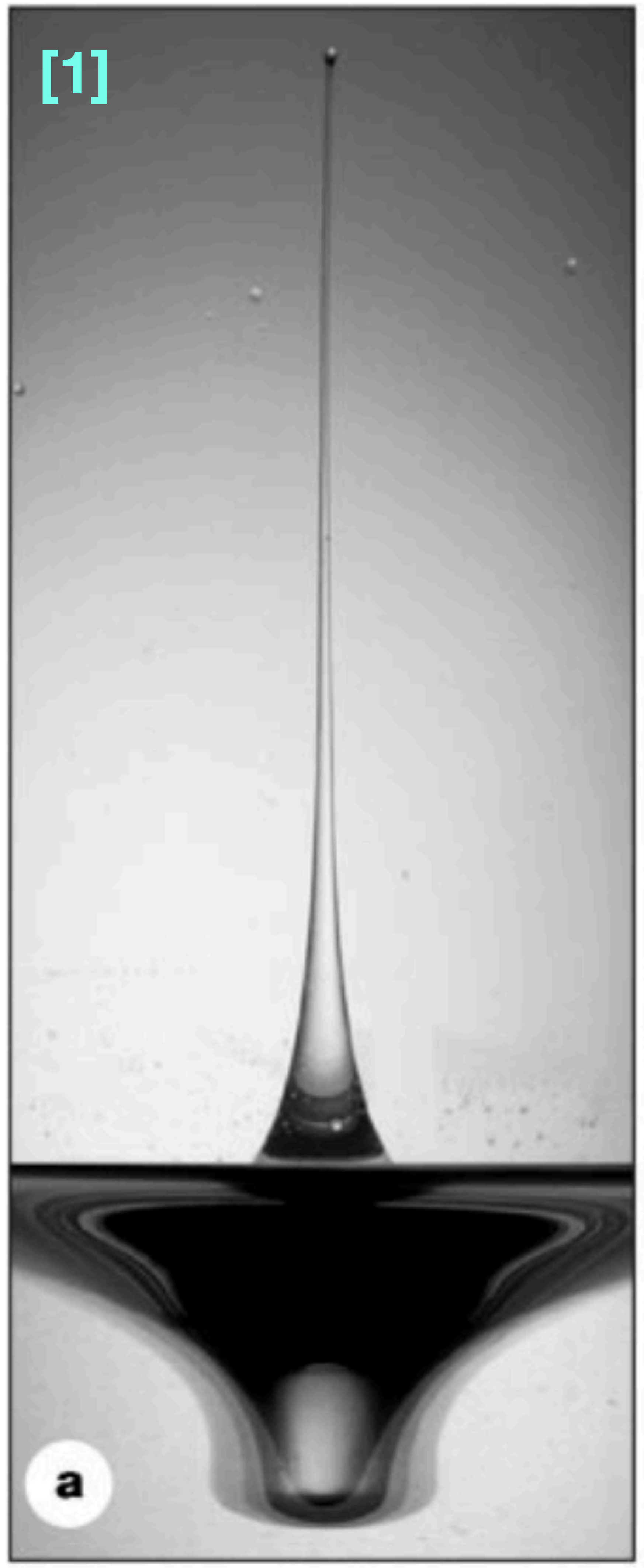
BASILISK (GERRIS) USERS' MEETING

* * *

5-7th July 2023

Experimental near-singular distortion of the free surface.

Balance between ***inertia*** *and* ***surface tension***.



Finite-time singularities
(collapsing Faraday Waves)

[1] Zeff *et al.*, *Singularity dynamics in curvature collapse and jet eruption on fluid surface*. Nature **403** (2000)

I.1 - Visualizing Scale Invariances (1/2)

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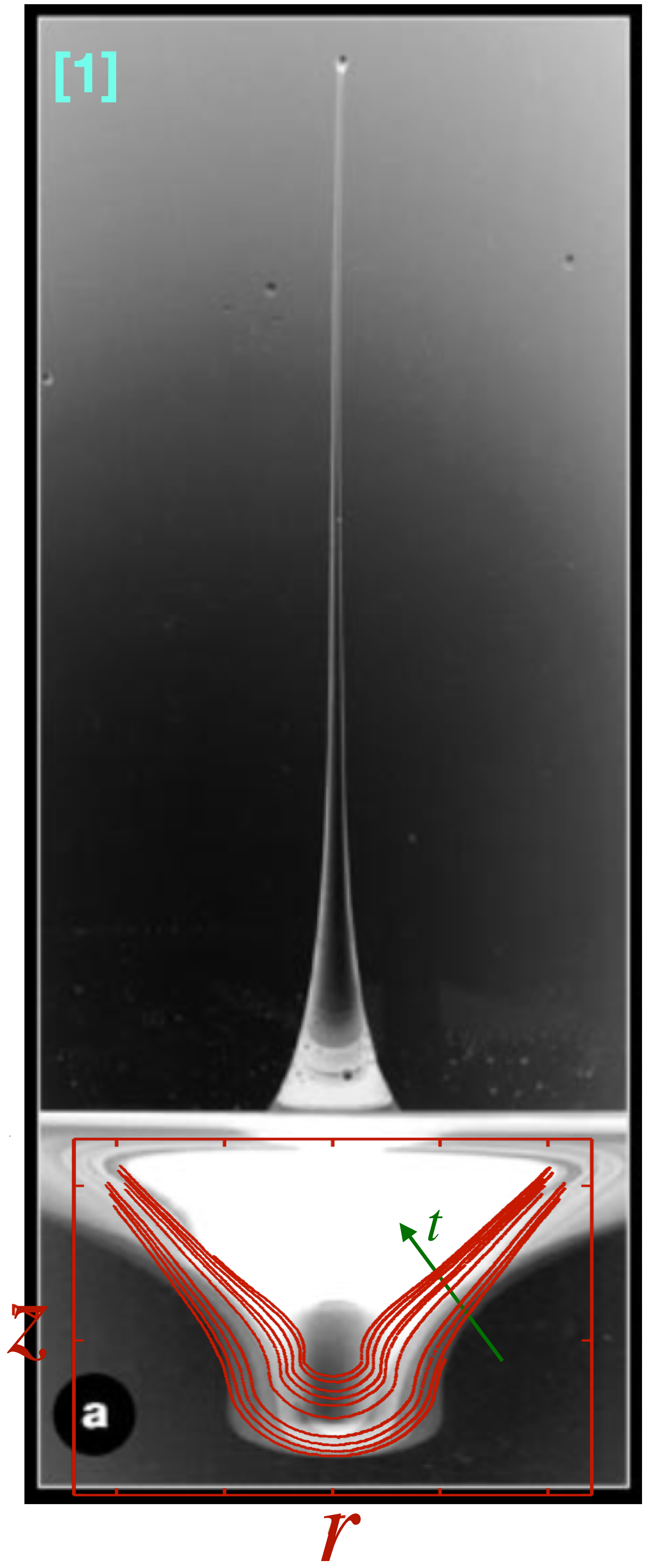
*Collapsing cavity profiles seem to have the **same shape**.*



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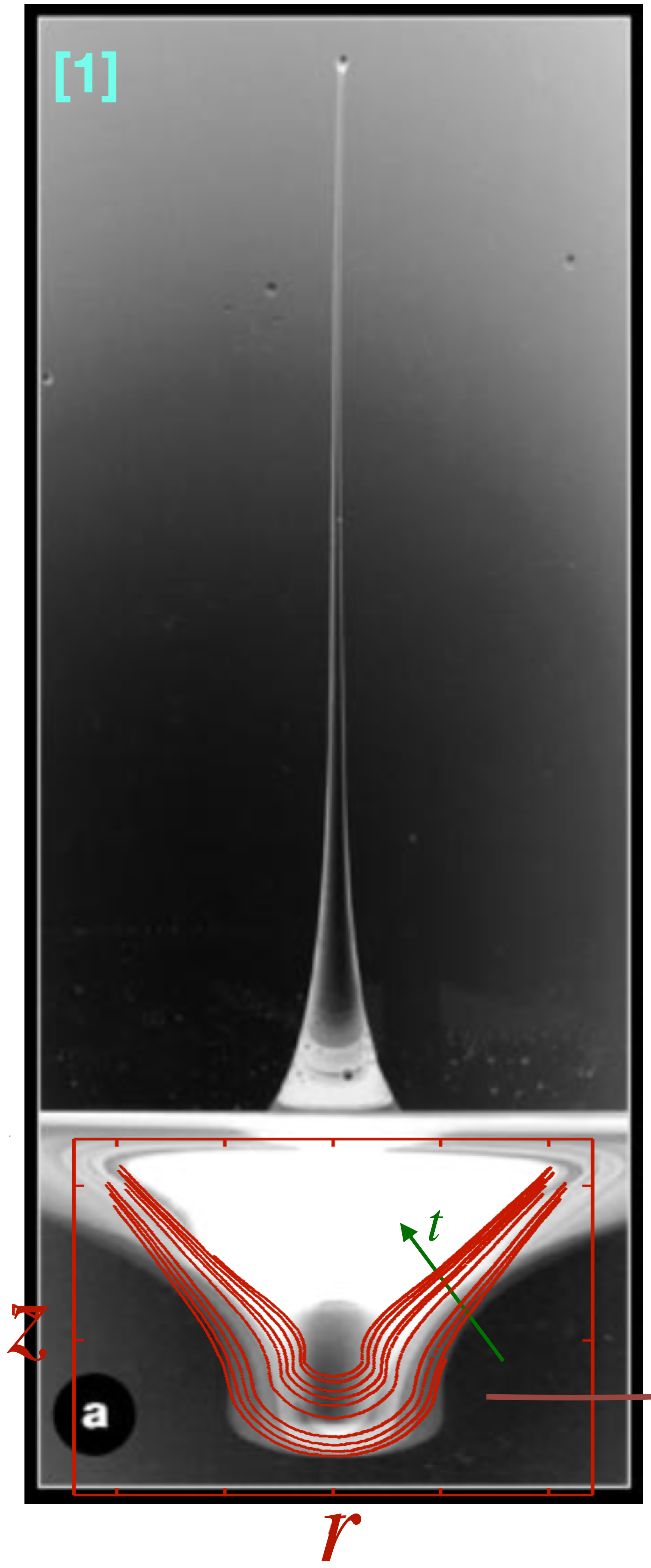
<u>Scale invariance</u> with *homotheties*



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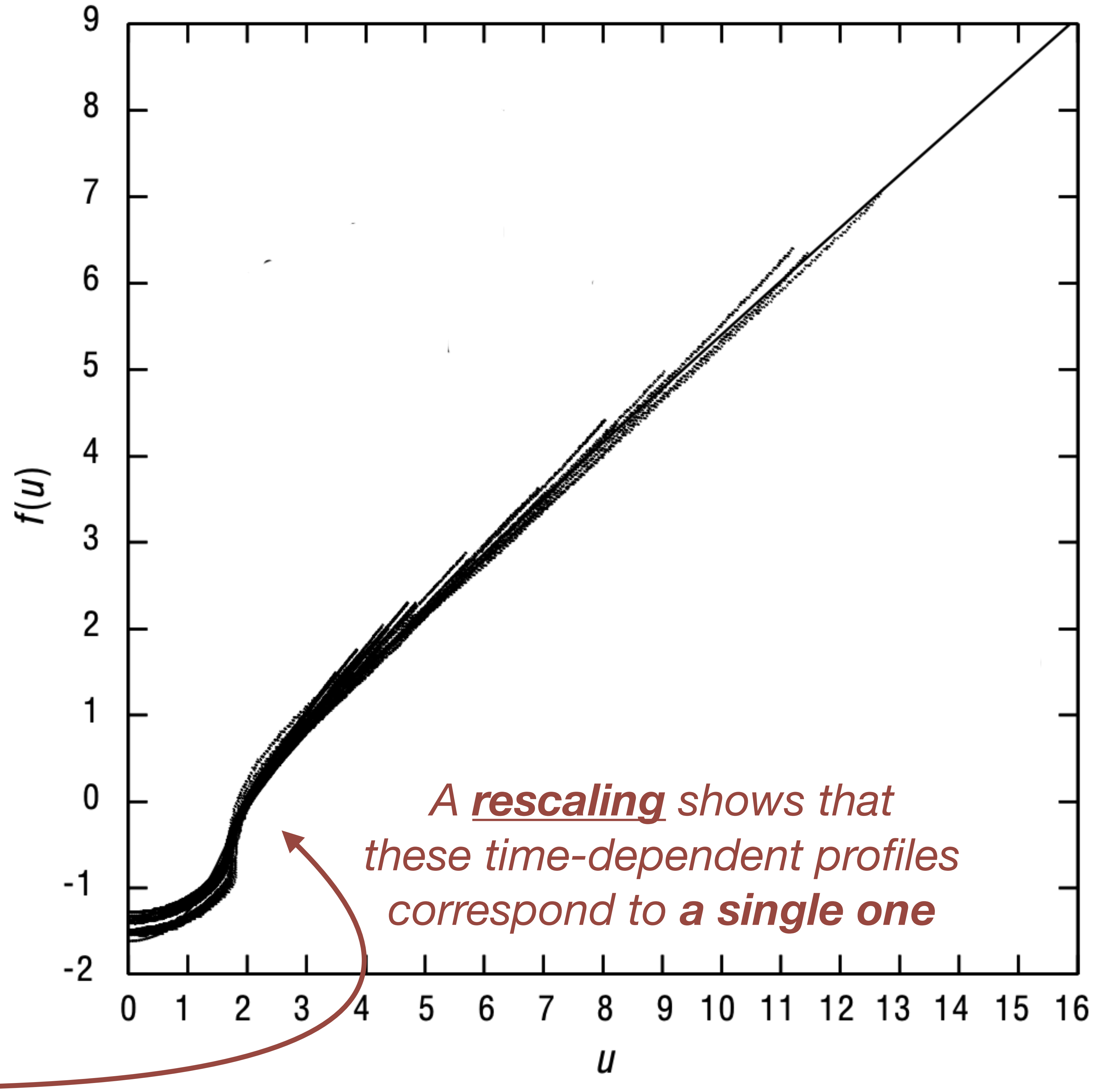

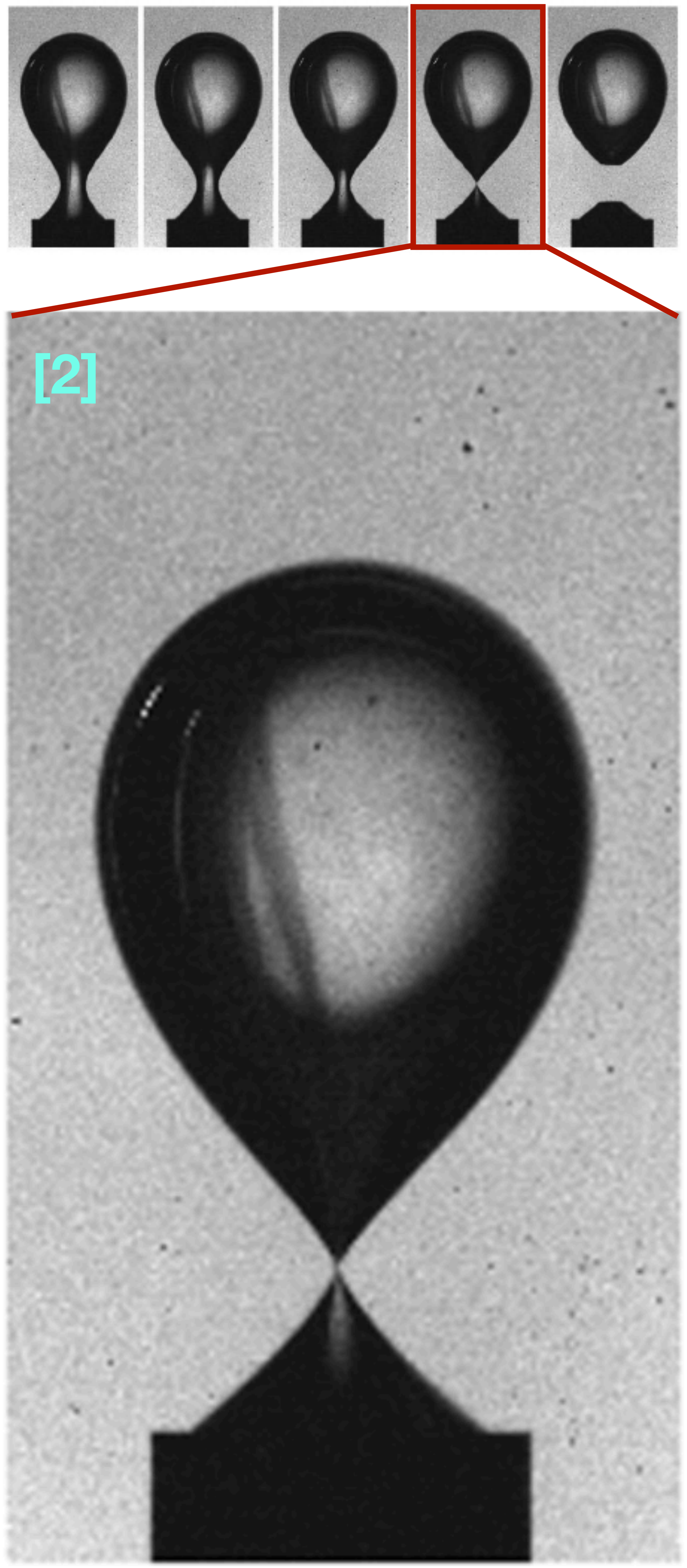

Bubble ***pinch-offs***

*A wide variety of **3D** physical problems highly **non-linear** involving locally **conical shapes**.*

Lengths scales evolving between 10^{-2} *and* 10^{-9} m.

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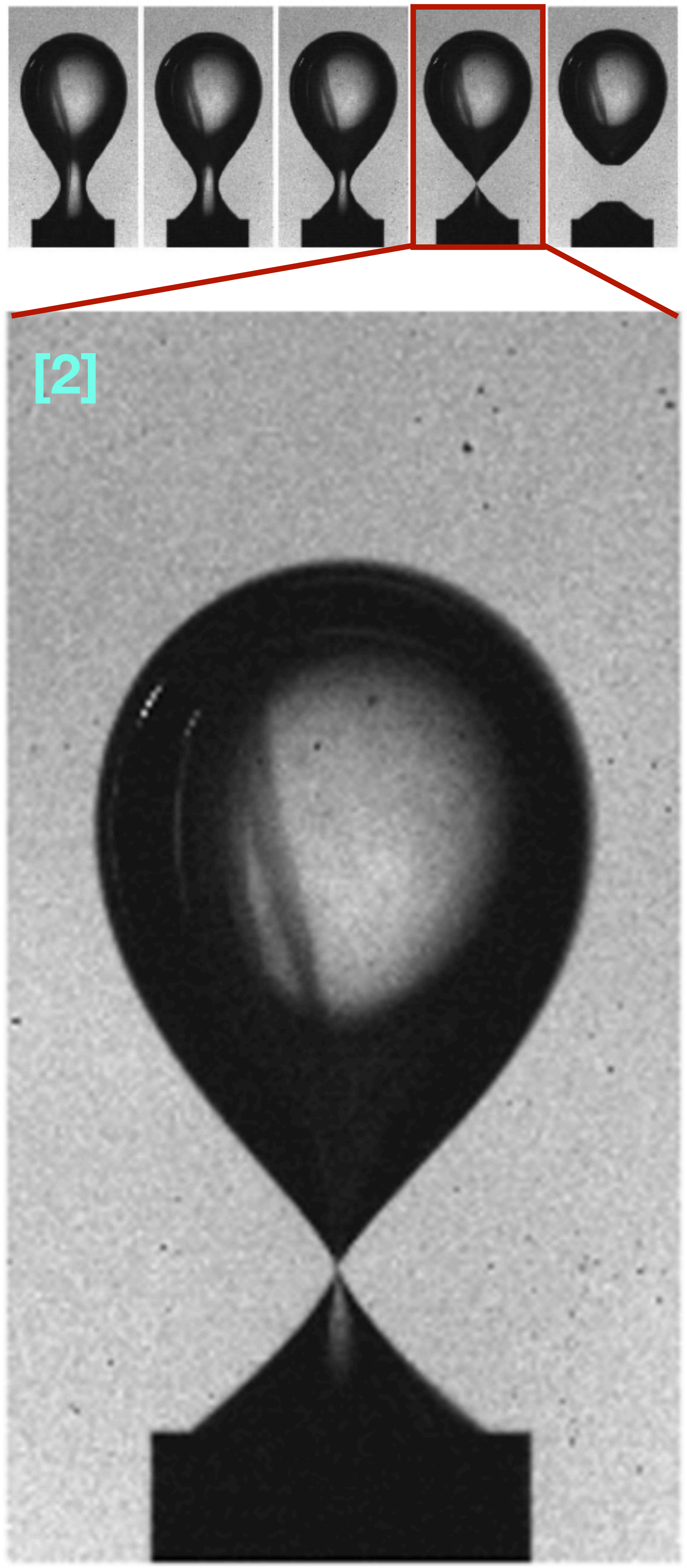

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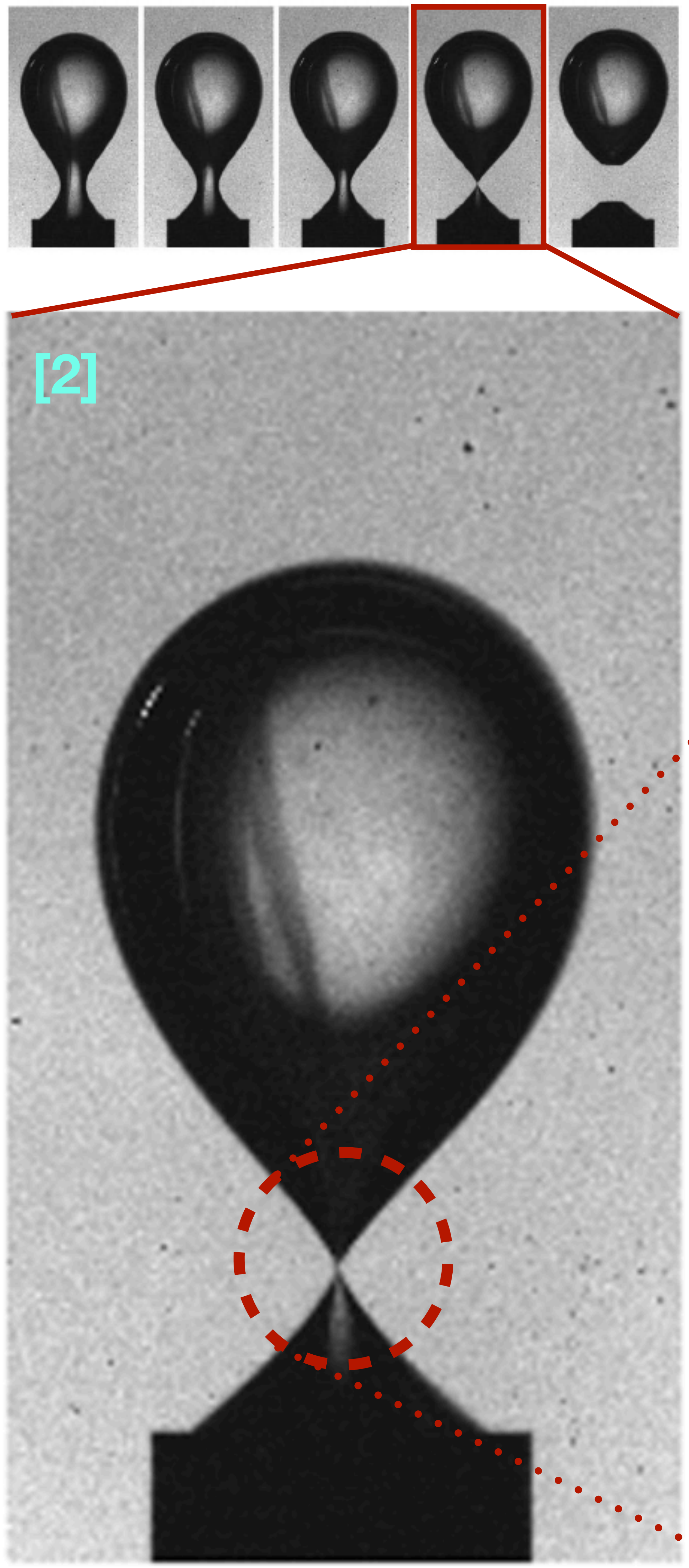
I - Introduction

I.1 - Visualizing Scale Invariances (2/2)



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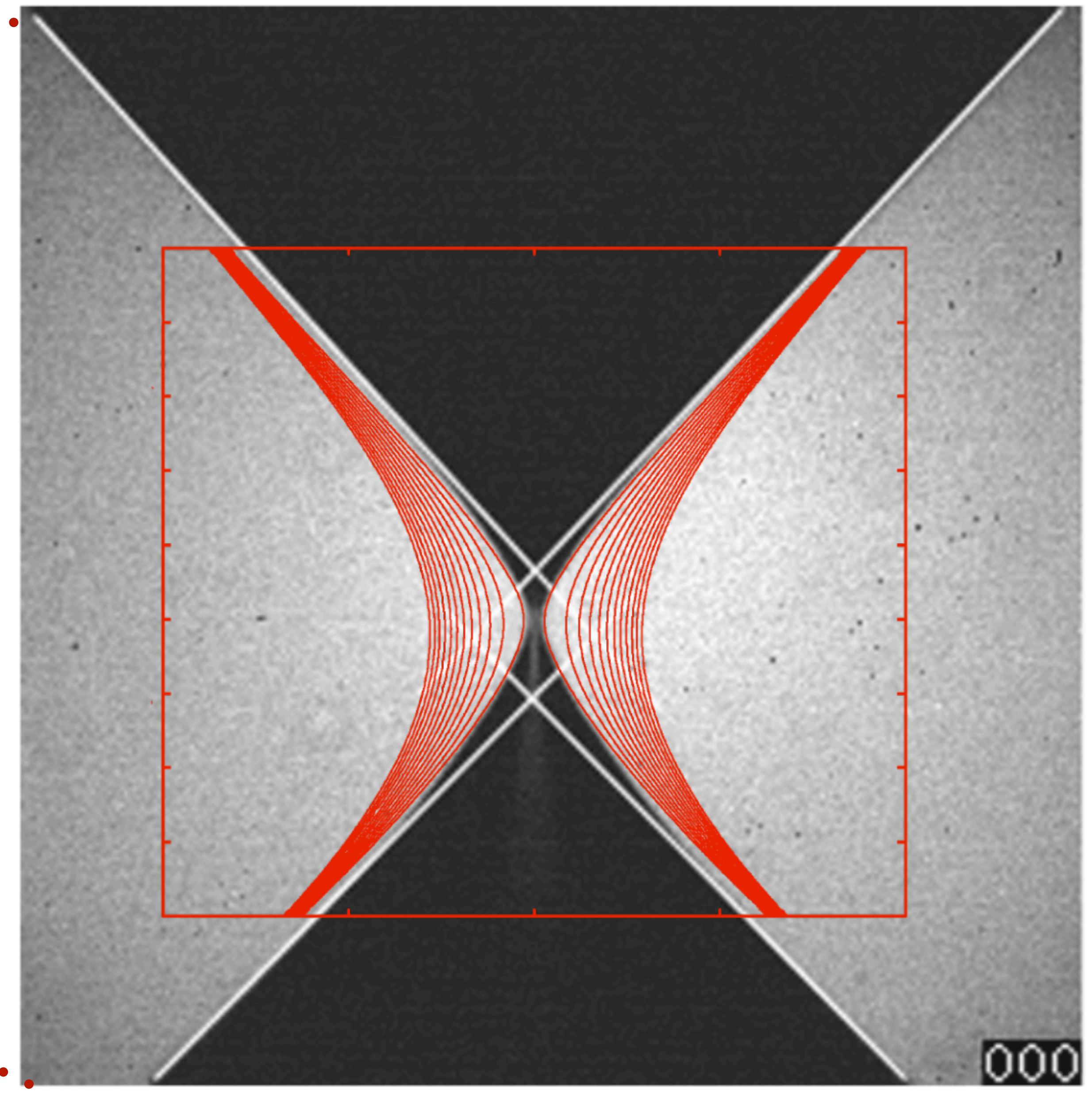

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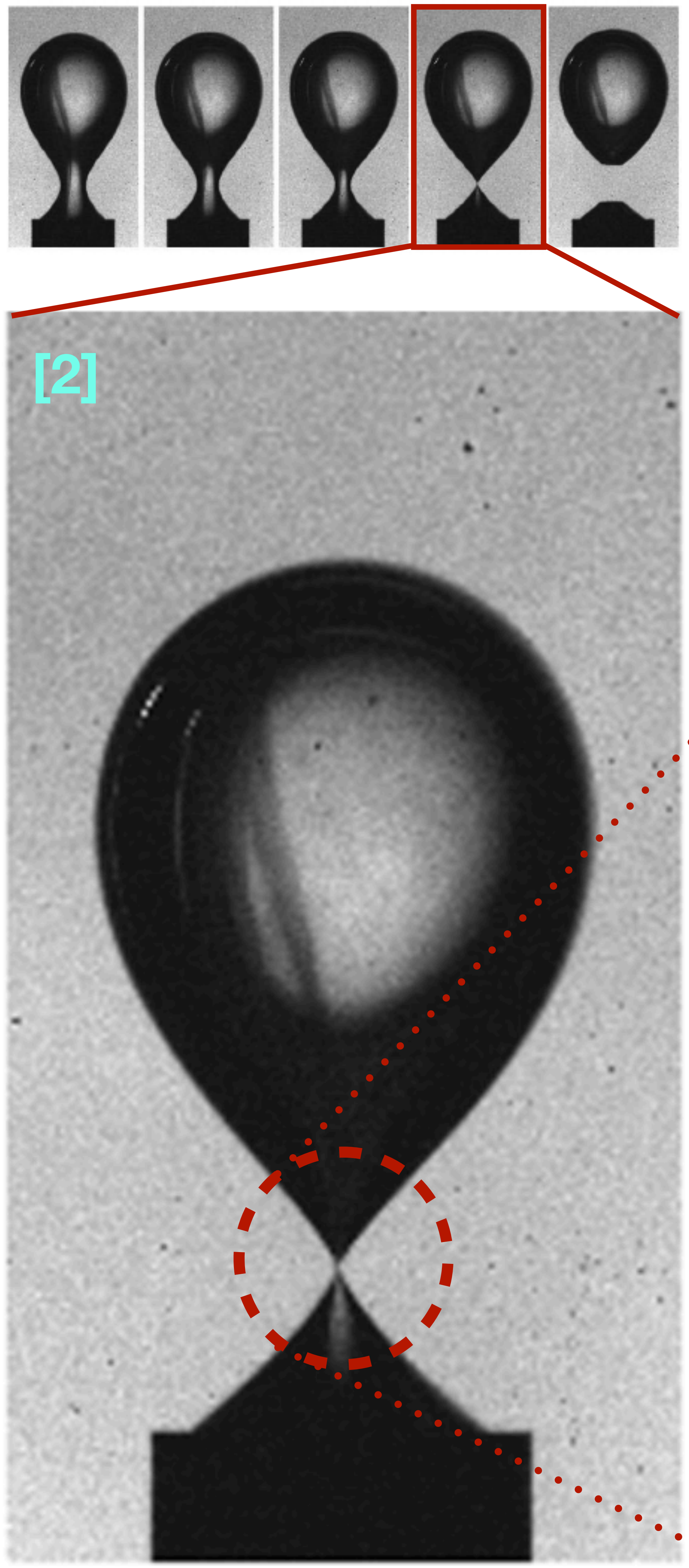

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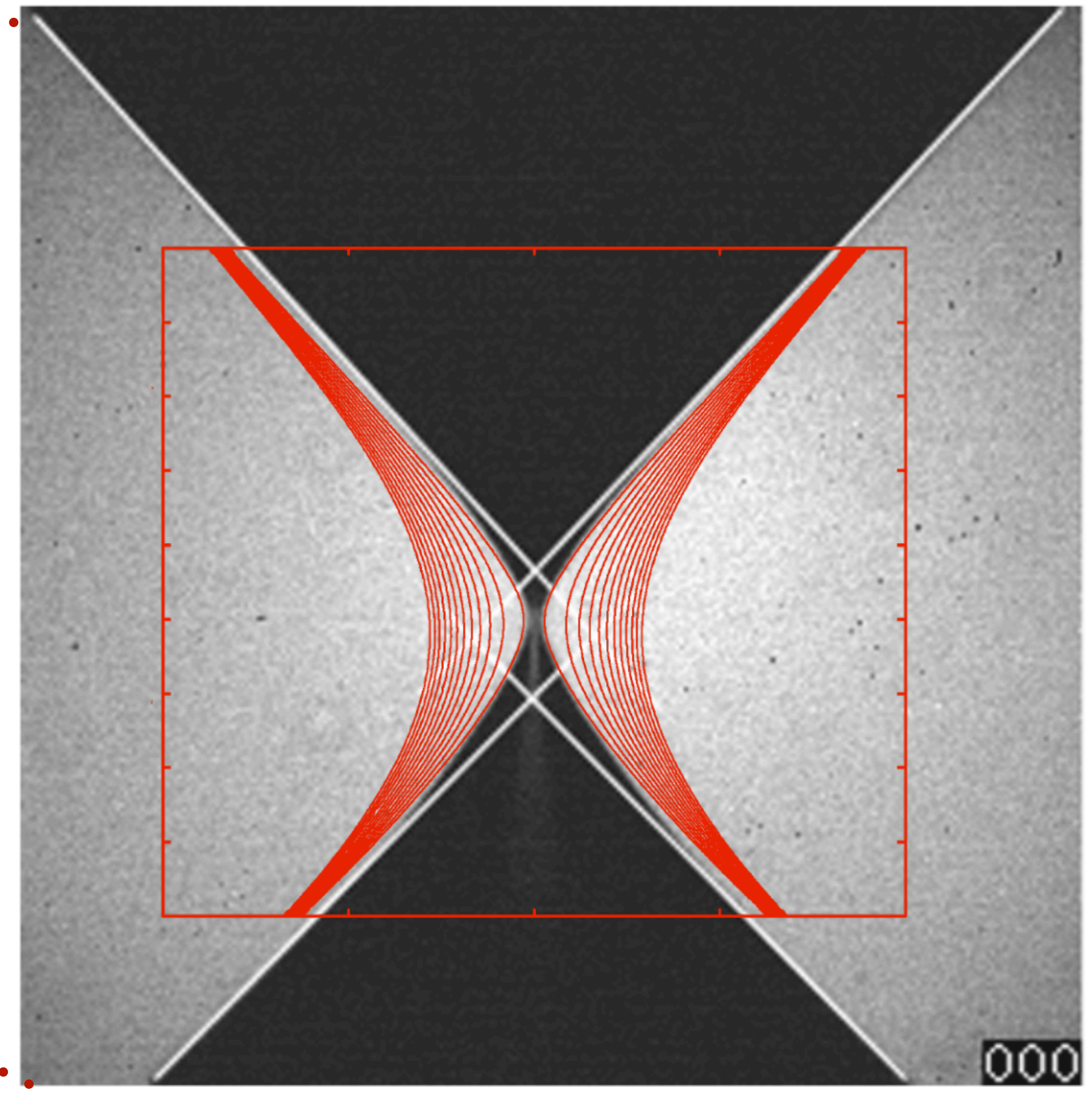

Bubble ***pinch-offs***

Again, scale invariance with cones seems to emerge, when looking at surface profile shapes!

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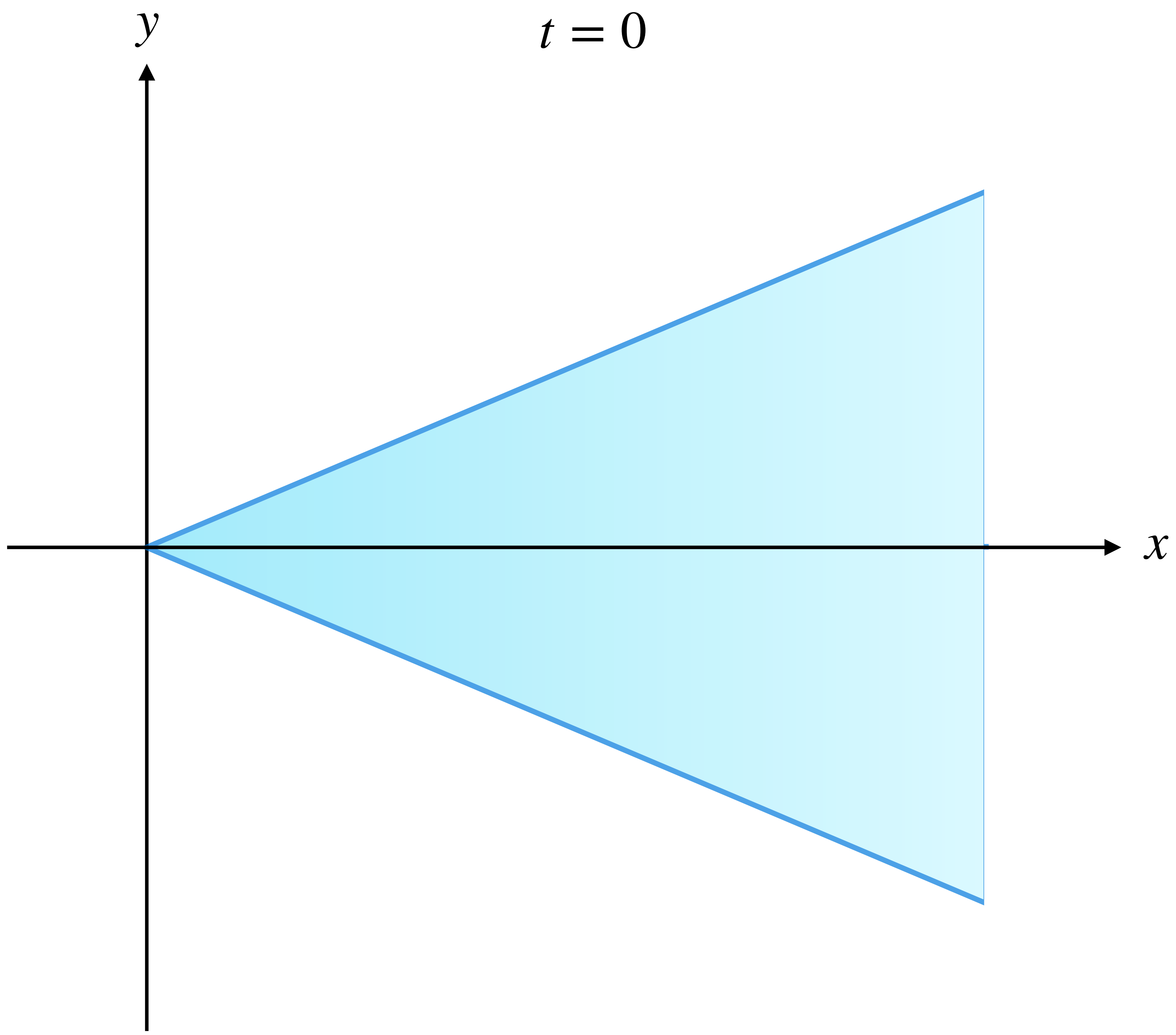
II.A - Keller & Miksis [5] problem

Breaking of a liquid sheet assimilated to a wedge.



Cross-section of a tapered sheet of liquid (wedge)

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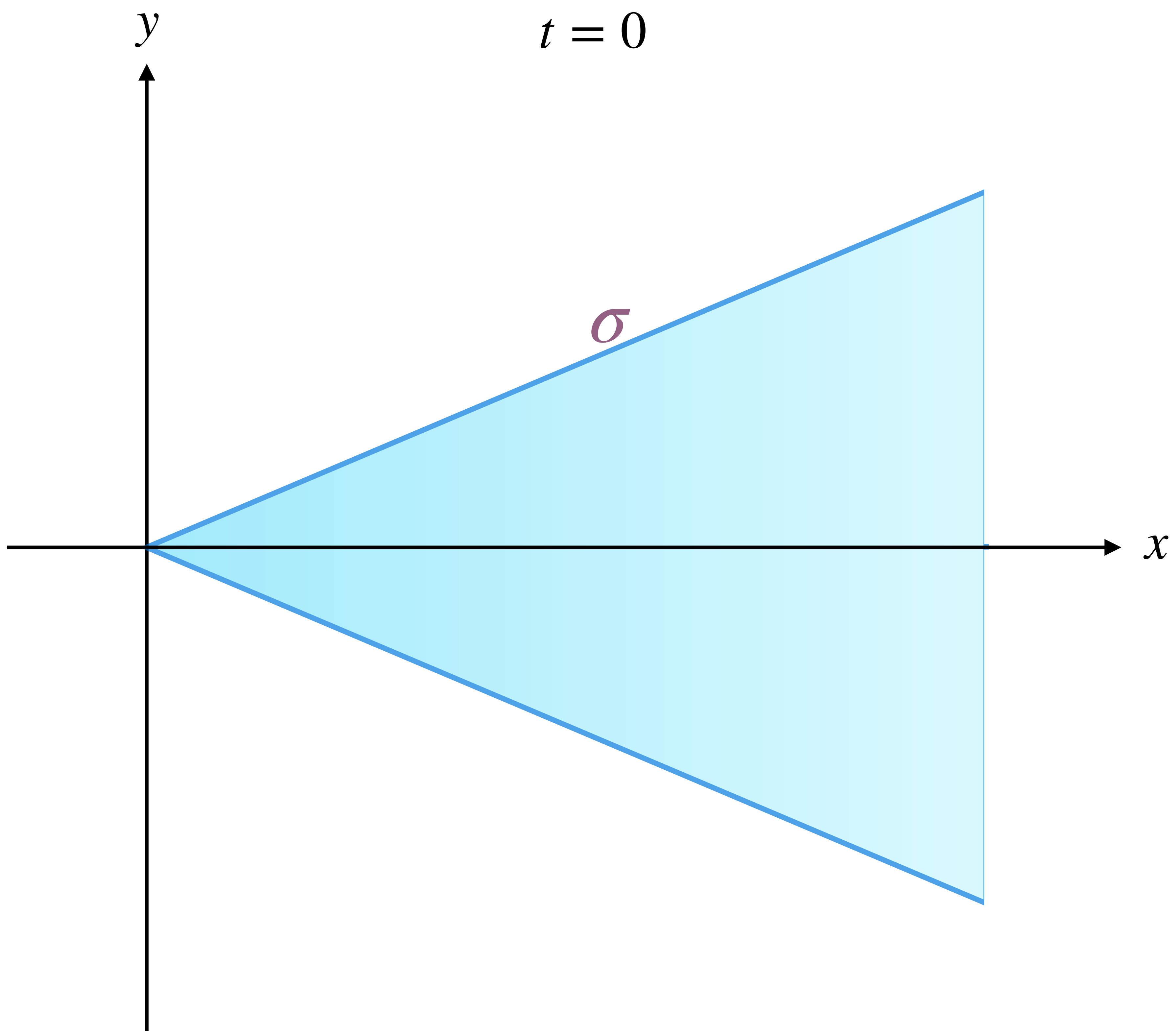

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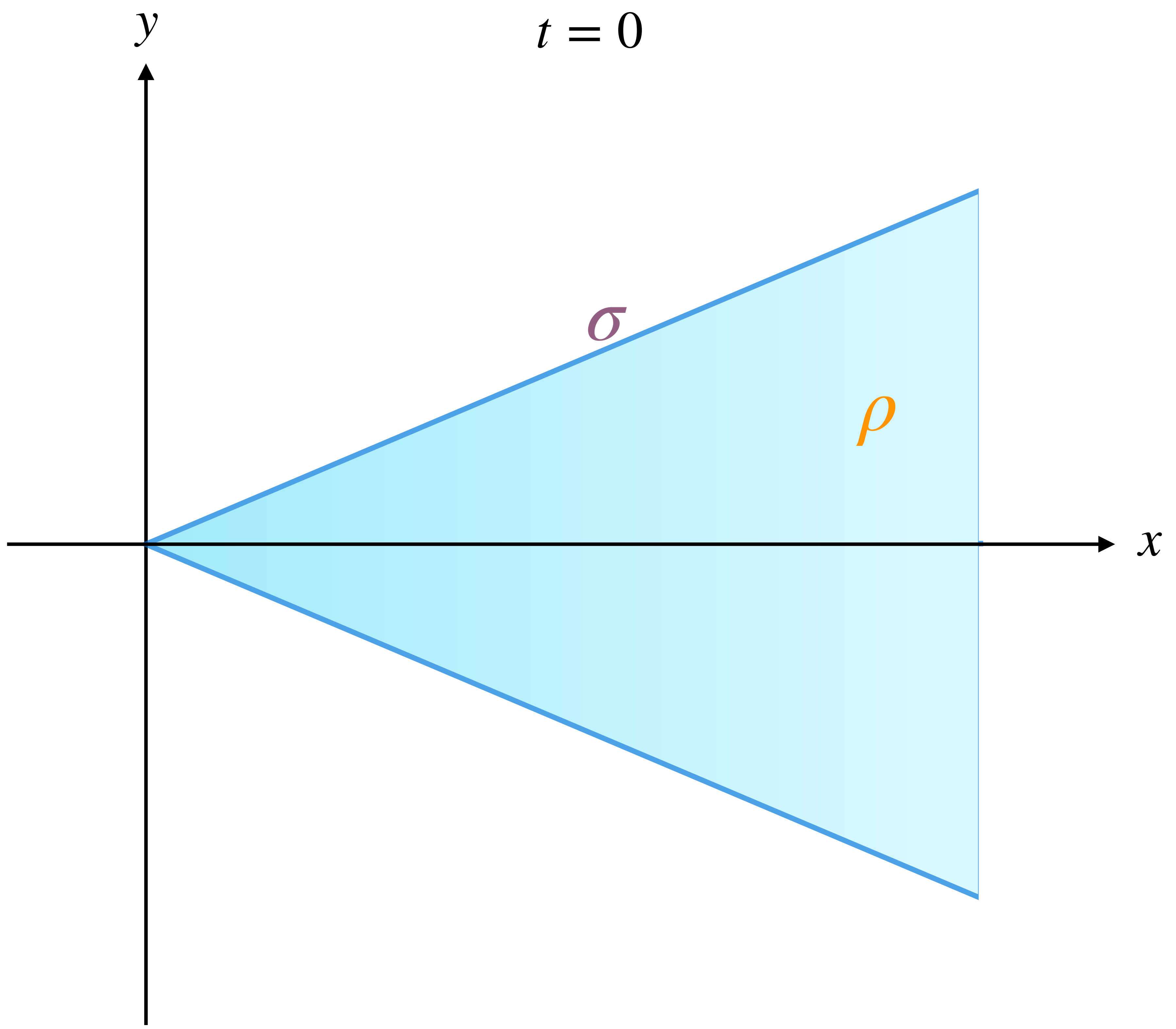

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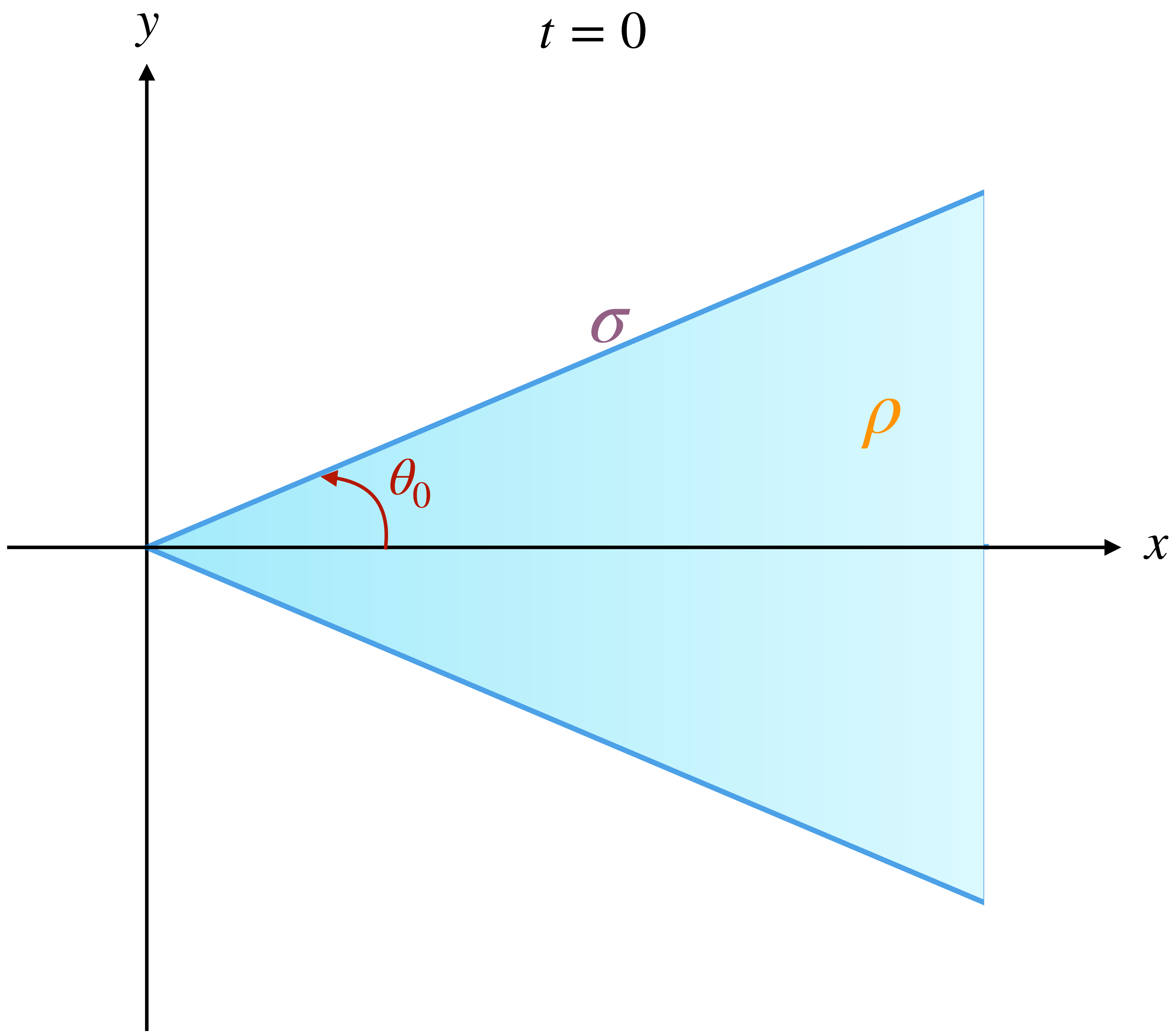

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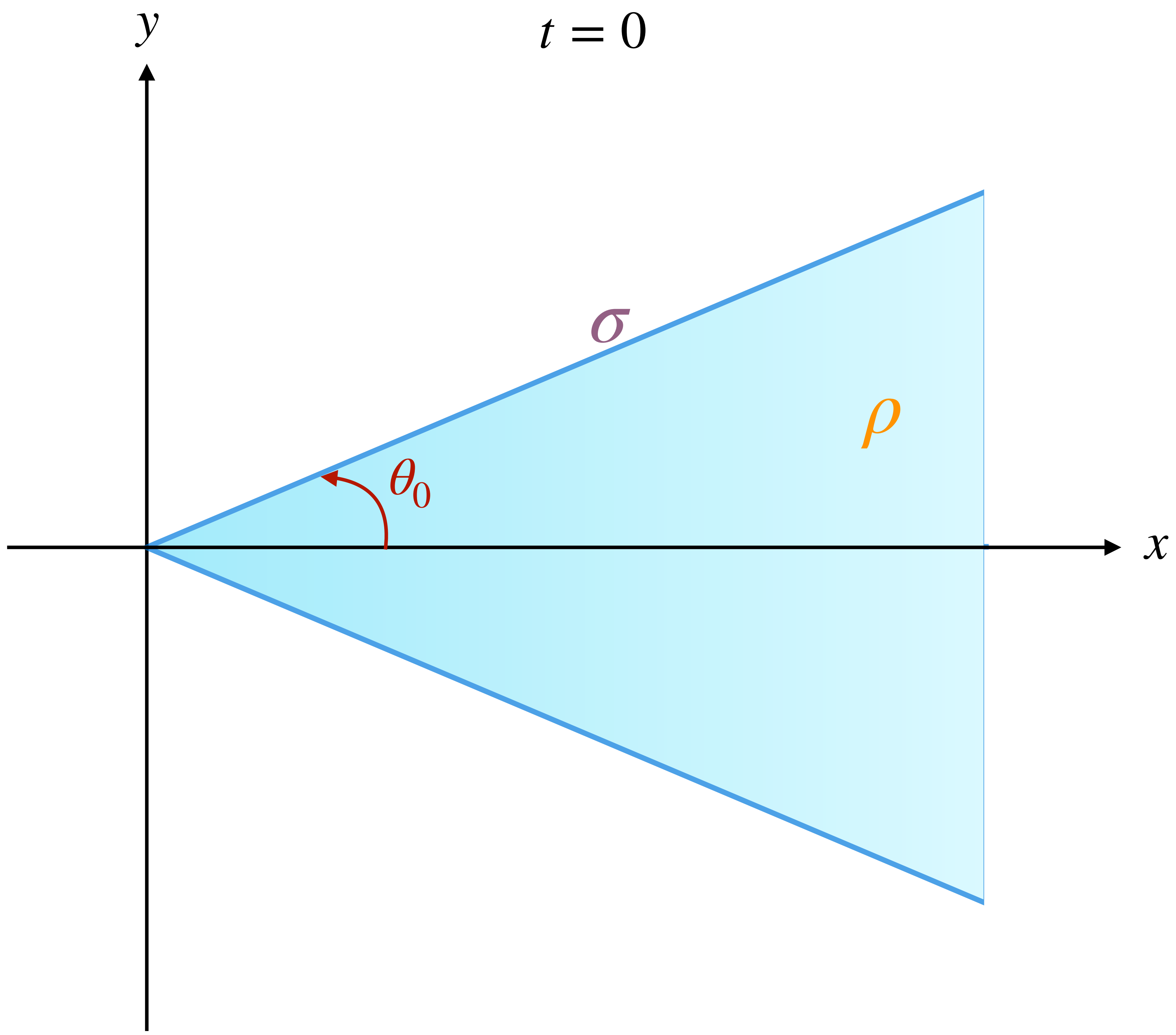

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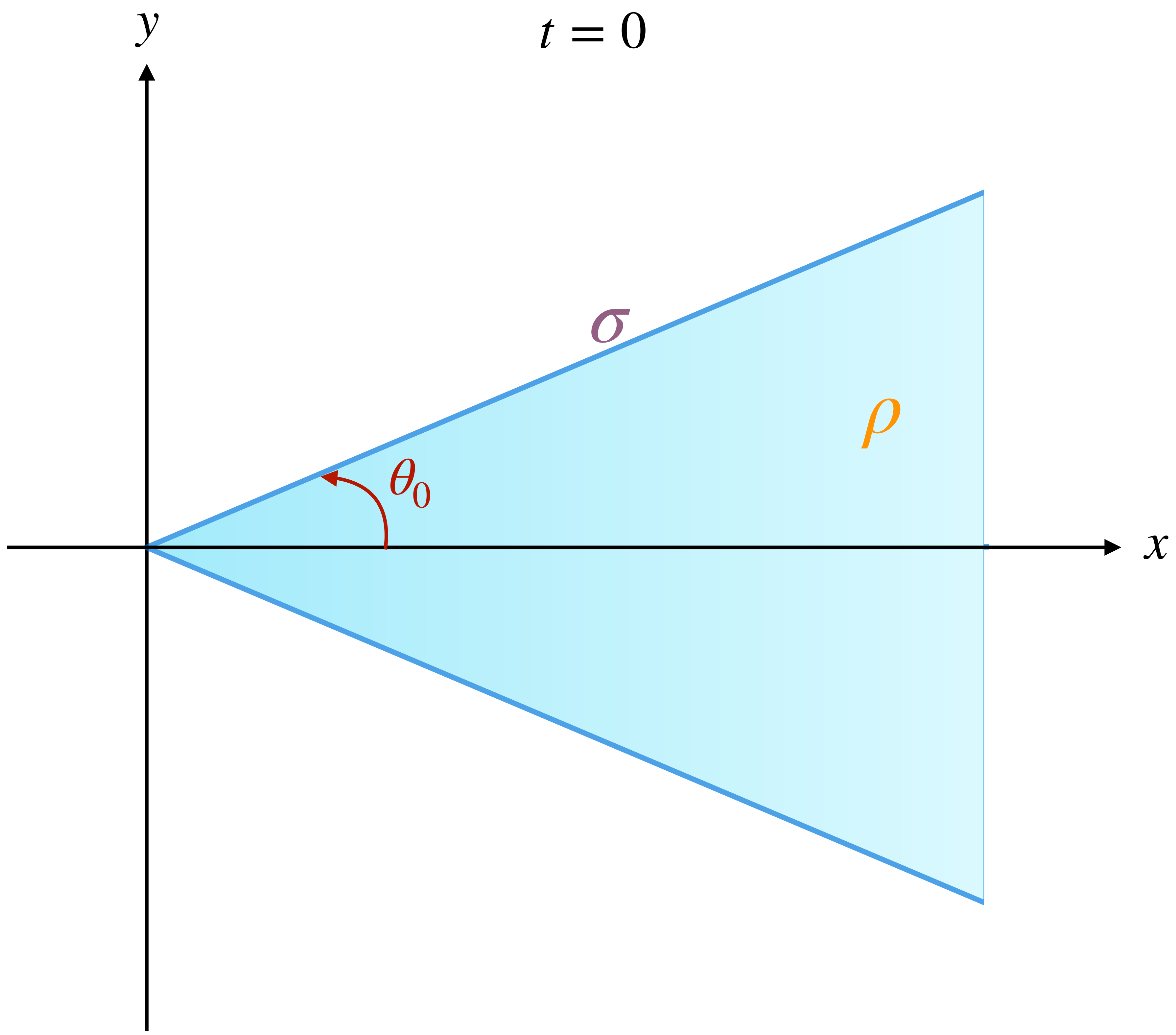
No viscosity taken into account.



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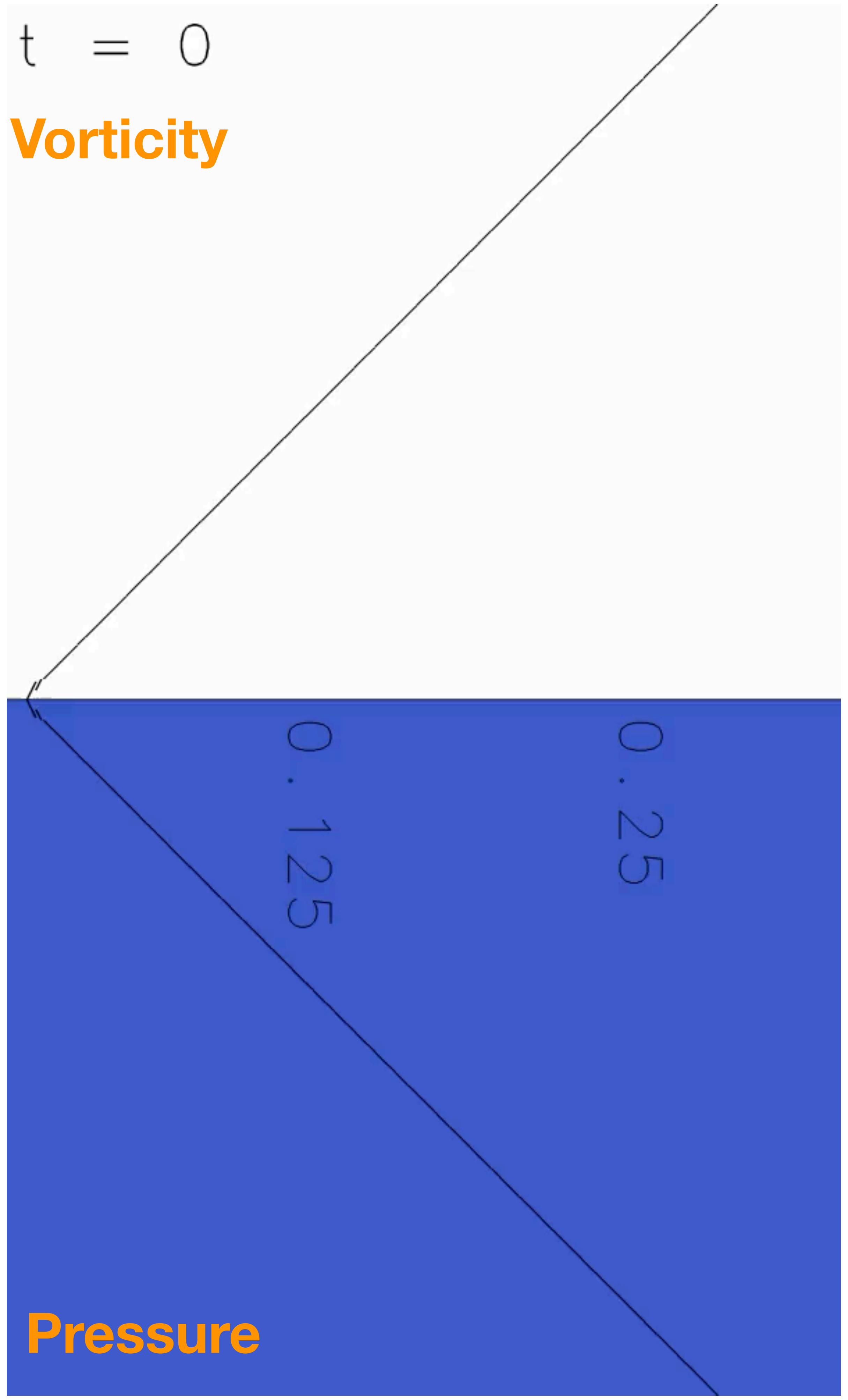
What is happening if we are simulating this study case?

$$N = 7$$
$$N_{max} = 8$$
$$L_0 = 1$$
$$\theta_0 = 45°$$
$$\beta_0 = 90°$$
$$\rho_{fluid} = 1$$
$$\rho_{gas} = 10^{-3}$$
$$\sigma = 1$$

Global Parameters



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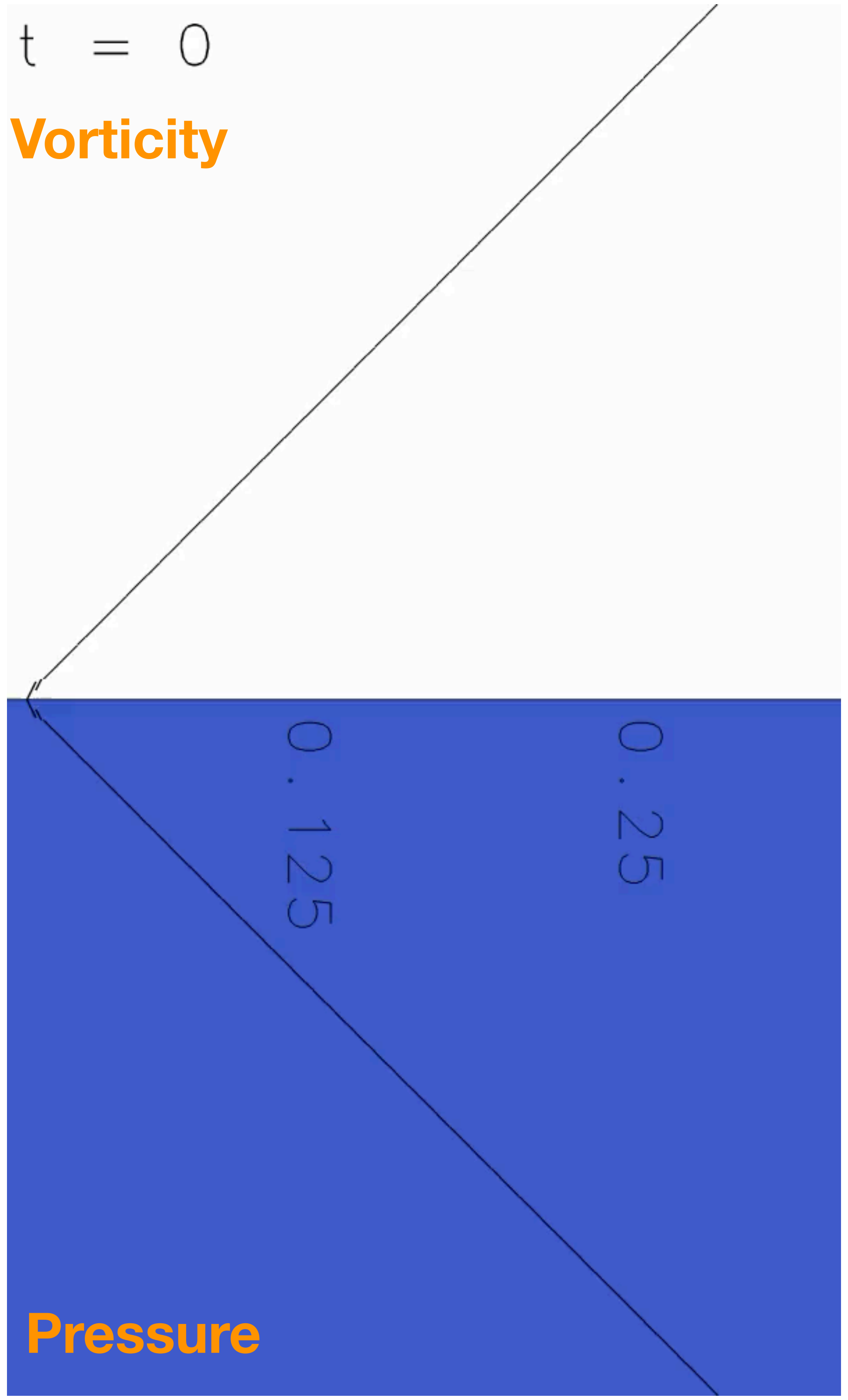

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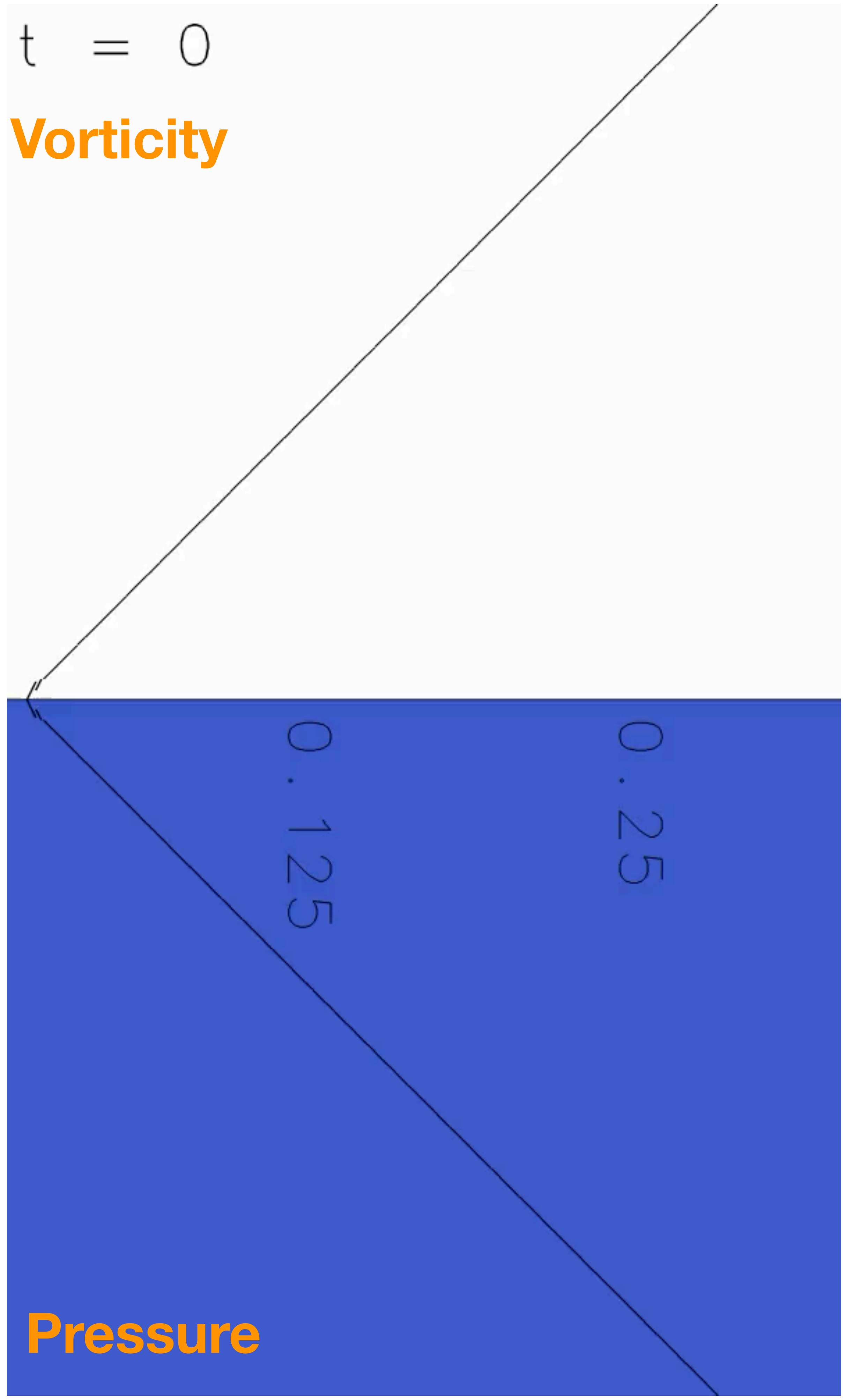

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II.2 - *Simulation in the Physical Space*

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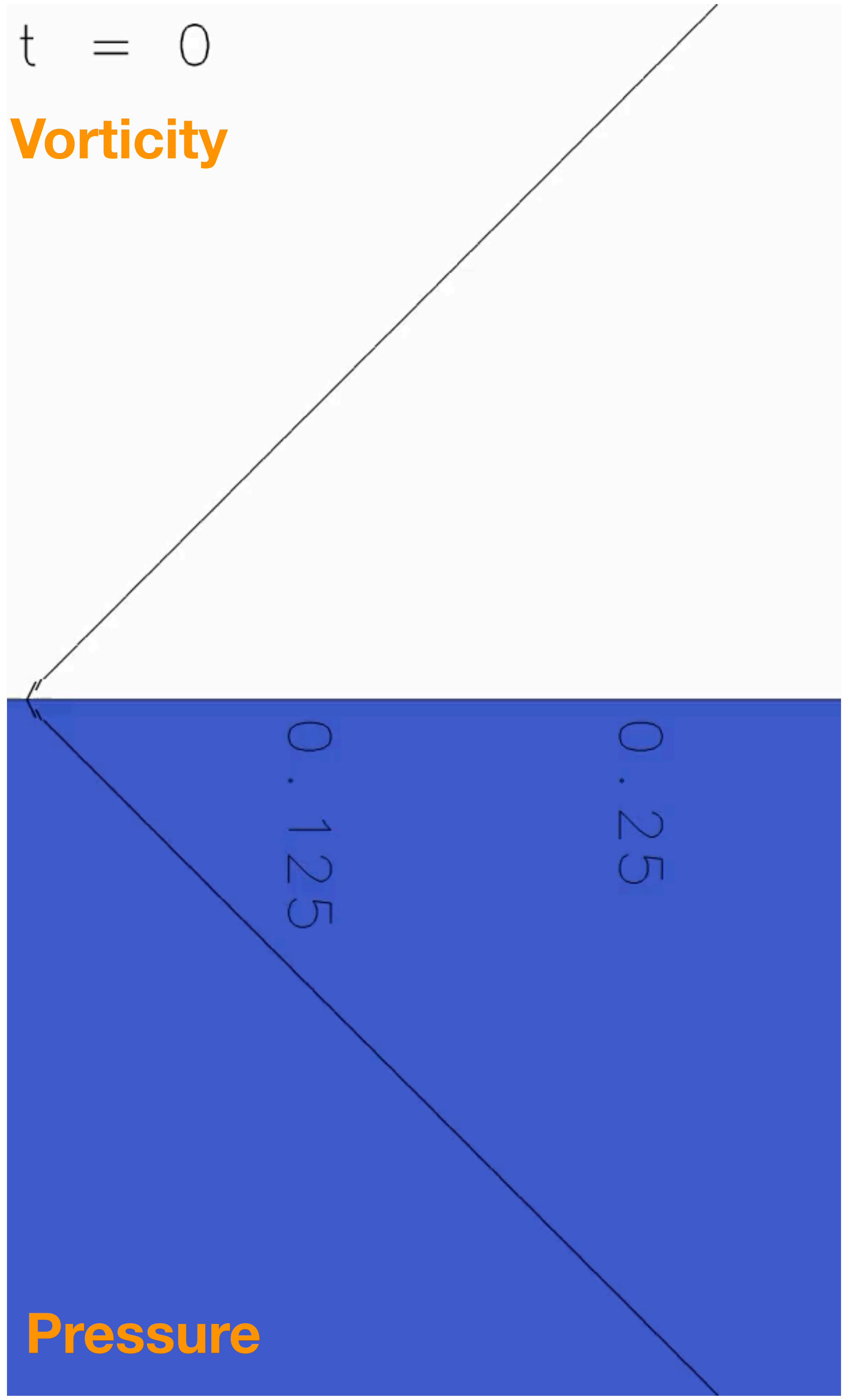

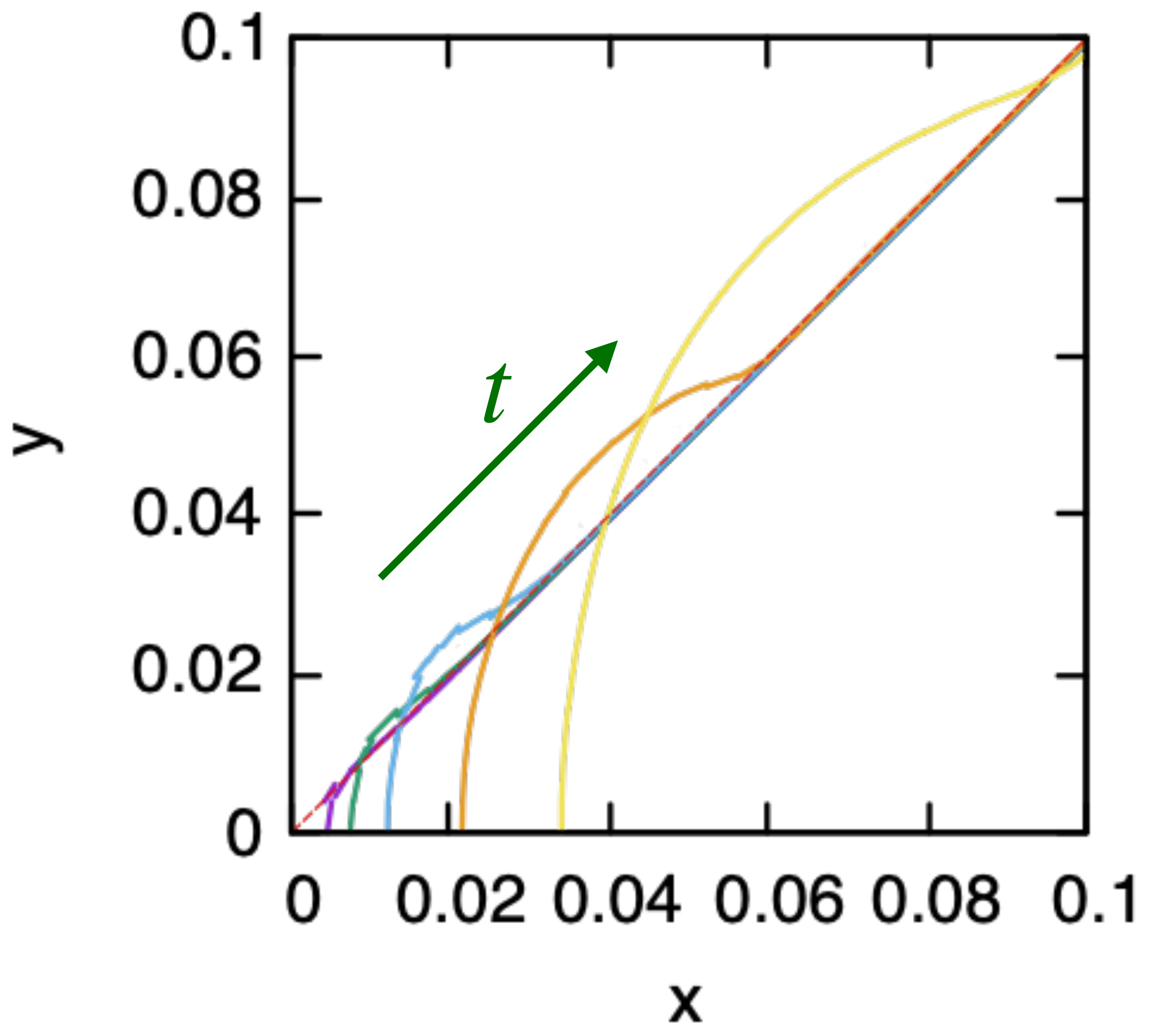

In the physical space, a *similar pattern* is observed, signature of a **scale invariance**?

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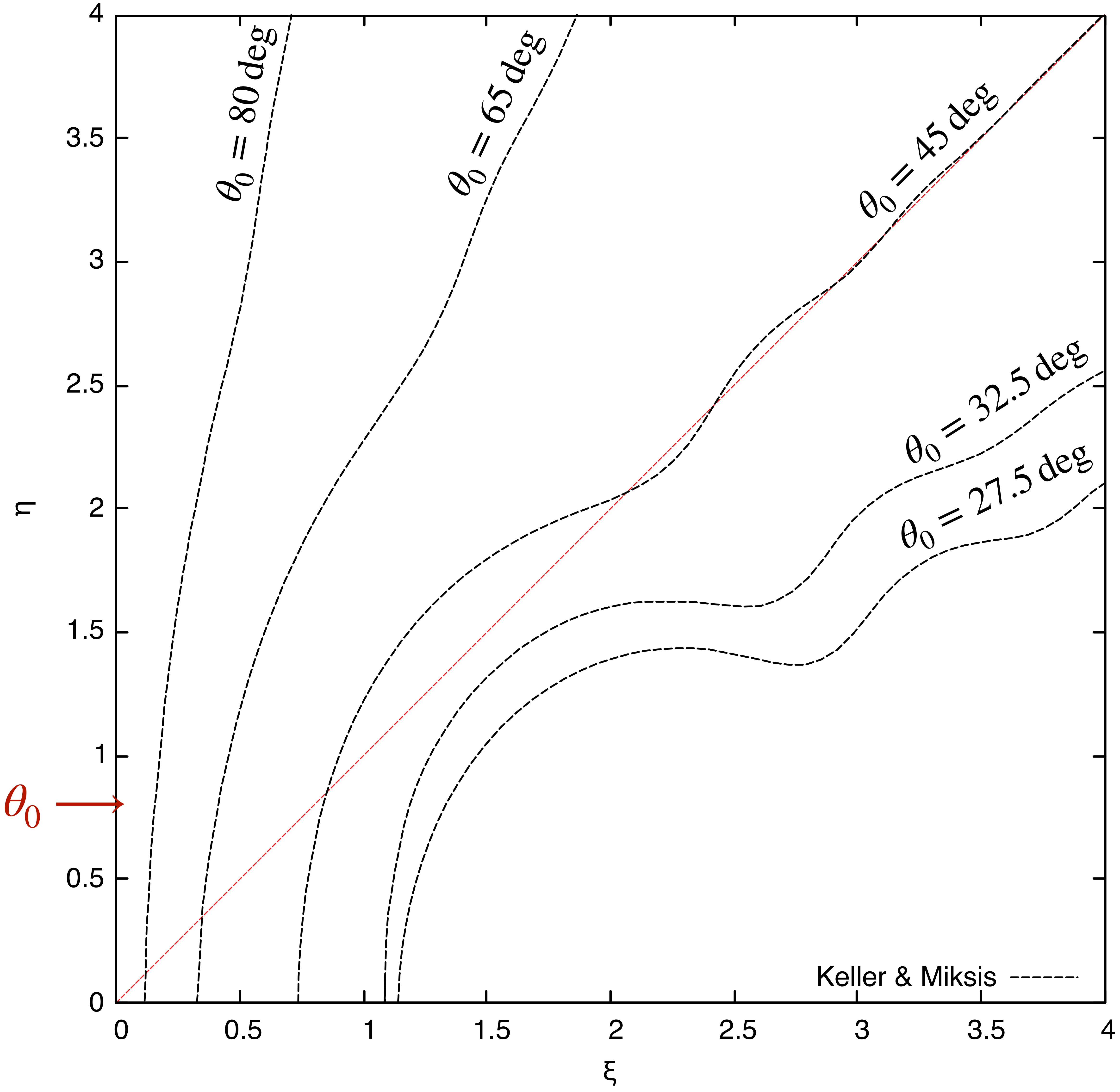
II.A - Keller & Miksis [5] problem

- Keller and Miksis determined theoretically the *scale invariance*
- Their numerical results were obtained:
  - by using a *Boundary Integral Method*;
  - searching for a **stationary solution**

To that end, they relied on 2 main assumptions:

1. The fluid is **irrotational**: *potential theory*
2. The fluid is **inviscid**.

STATIONARY SOLUTIONS FOR DIFFERENT WEDGE ANGLES θ_0 →



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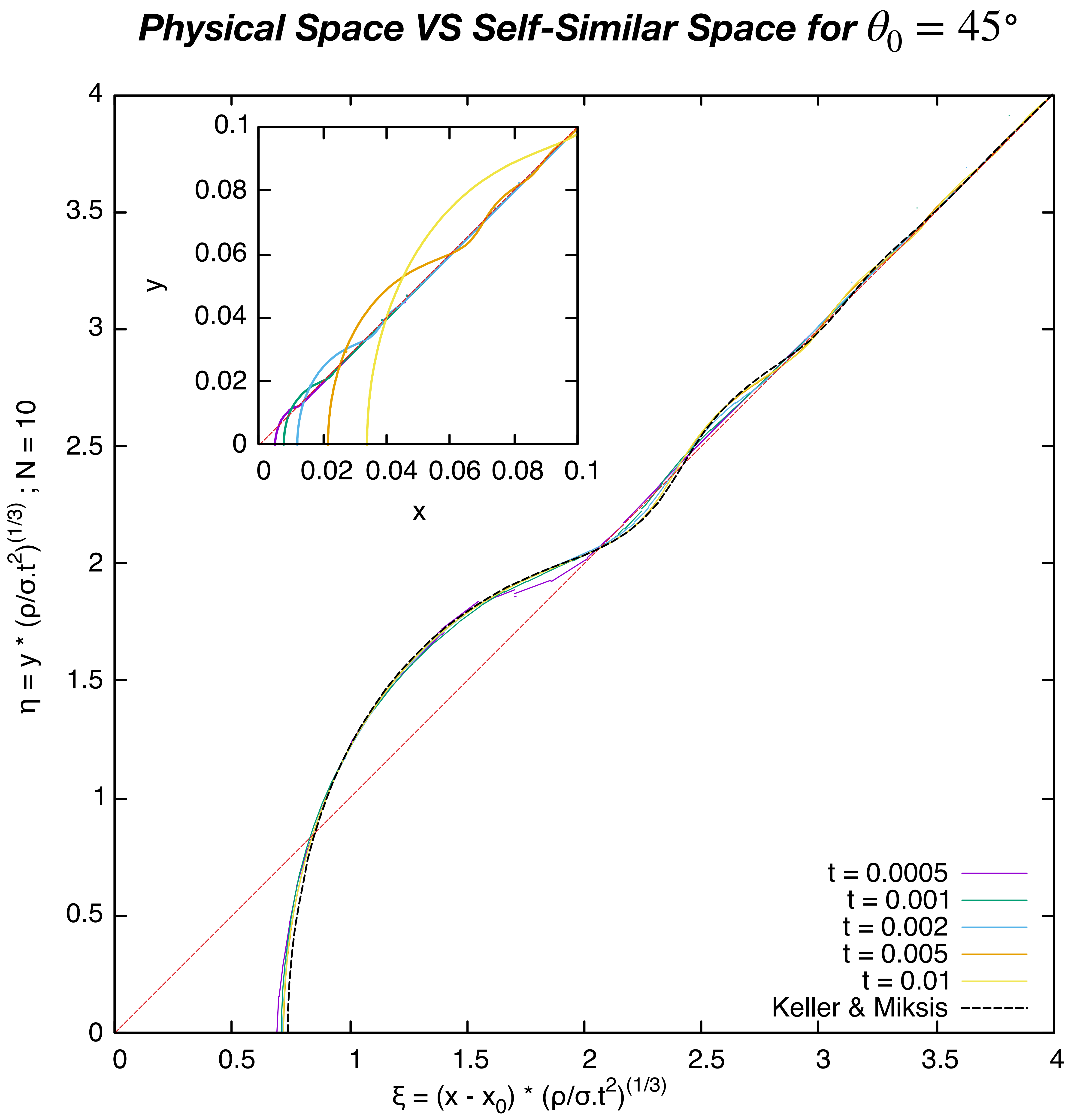
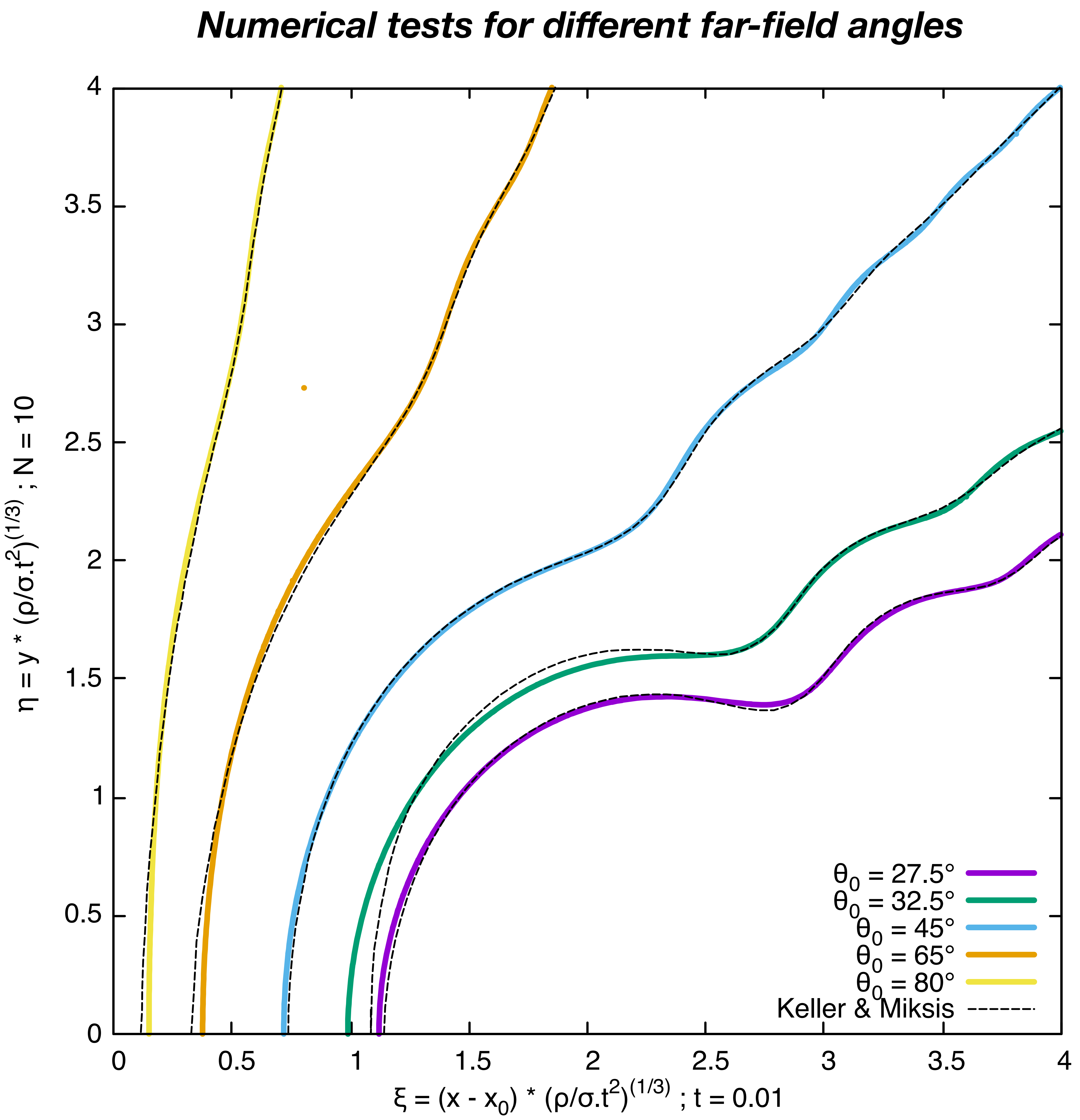


II.4 - Rescaled simulations: invariance



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   After changing scales, the problem written for x_i' is formally identical to the one written for x_i; this implies ***constraints***.

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3. **Choice of free parameters:**
   Counting the number of *independent relationships* R and *scaling factors* S:
   $F = R - S$ free parameters letting invariant the problem.

II.5 - The Scale Invariance Method

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4. **Consequences for the solution:**
   As the original problem is invariant to these changes of scale, so is its solution.

Simplified cartesian Navier-Stokes eqns:

$$\partial_x u = -\,\partial_y v$$

$$\partial_t u + u\,\partial_x u + v\,\partial_y u = -\,\partial_x(p/\rho)$$

$$\partial_t v + u\,\partial_x v + v\,\partial_y v = -\,\partial_y(p/\rho)$$

Initial Condition (free-surface shape):

$$f(x, y, 0) = y\cot\theta_0 - x$$

Kinematic Condition:

$$\partial_t f + u\partial_x f = 0$$

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Dynamic Condition:

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where κ is the *curvature.*

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$$\frac{u^*}{v^*} \overset{1}{=} \frac{x^*}{y^*} \qquad \frac{u^*}{t^*} \overset{2}{=} \frac{u^{*2}}{x^*} \overset{3}{=} \frac{1}{x^*}\frac{p^*}{\rho^*} \qquad p^* \overset{4}{=} \frac{\sigma^*}{x^*} \qquad x^* \overset{5}{=} y^*$$

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8 scaling factors
5 independent relations $\Rightarrow$ **3 *free param.*** $\Rightarrow$ (σ^*, ρ^*, t^*)

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8 scaling factors
5 independent relations $\Rightarrow$ **3 *free param.*** $\Rightarrow$ (σ^*, ρ^*, t^*)

Variables can be expressed according to this choice of parameters:

$$x^* = y^* = \left(\frac{\sigma^* t^{*2}}{\rho^*}\right)^{1/3} \qquad u^* = v^* = \left(\frac{\sigma^*}{\rho^* t^*}\right)^{1/3} \qquad p^* = \rho^* \left(\frac{\sigma^*}{\rho^* t^*}\right)^{2/3}$$

III.1 - Motivation for computing directly into the Self-Similar Space

- **Self-similar solution:** *stationary* solution in a particular rescaled space, thanks to the study of scale invariances throughout the physical problem

- Previous slides: only *rescaling* ⇒ all the needed scales have to be simulated

- Simulate directly into the self-similar space = all the scales have become $O(1)$
  ⇒ overcoming numerous numerical obstacles (*meshes*)

- To include easily **viscosity** effects:
  not done in papers **[5]**, **[6]**, **[7]** using *potential theory*

[5] Keller and Miksis, *Surface Tension Driven Flows*. SIAM **43** (1983)

[6] Day *et al.*, *Self-Similar Capillary Pinchoff of an Inviscid Fluid*. PRL **80** (1998)

[7] Sierou and Lister, *Self-similar recoil of inviscid drops*. Physics of Fluids **19** (2004)

- We search for the most general solution to the *N-S.* equations while using **scale invariant variables**, under the form $f\left(\tilde{X}, \tilde{Y}, \tau\right)$, where:

$$\tau = \ln t$$

following **[8] (*logarithmic time*).**

III - Self-Similar Navier-Stokes solver with *Basilisk*

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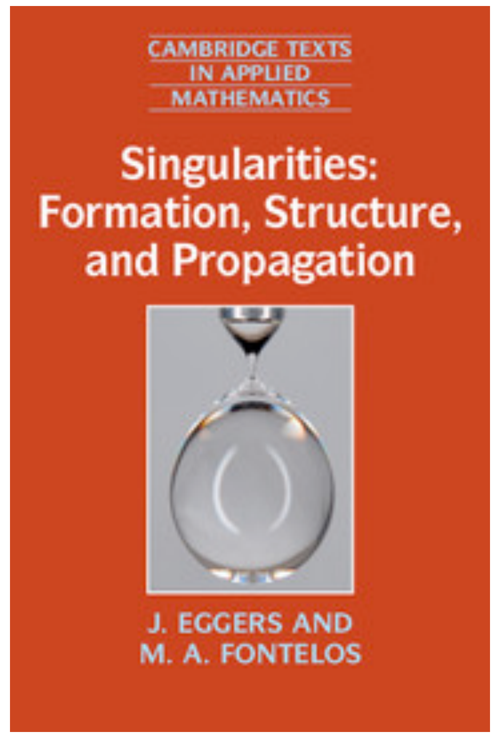


[8] Eggers and Fontelos, *Singularities: Formation, Structure, and Propagation.* Cambridge Texts of Applied Mathematics (2015)

III - Self-Similar Navier-Stokes solver with *Basilisk*

III.2 - Strategy for finding S-I. Solutions

- We search for the most general solution to the *N-S.* equations while using **<u>scale invariant variables</u>**, under the form $f\left(\tilde{X}, \tilde{Y}, \tau\right)$, where:

$$\tau = \ln t$$

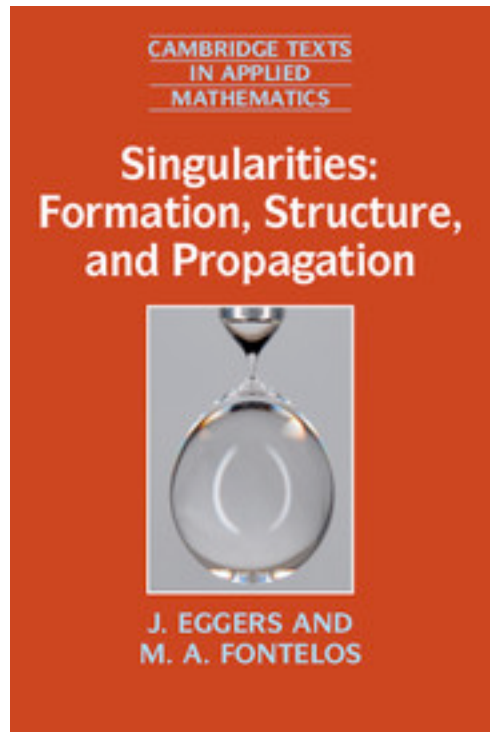
following **[8] (*logarithmic time*).**

- The introduction of the variable τ:

  $\Rightarrow$ gives an *evolution equation* **<u>in</u>** the self-similar space, solved with *Basilisk*;

  $\Rightarrow$ while the problem remains *unsteady*, the solution is <u>NOT</u> *scale invariant*;

  $\Rightarrow$ *Basilisk* is used to study **<u>how</u>** the numerical solution converges or not towards a *scale invariant solution*, *i.e.* a solution when $\partial_\tau f \to 0$ condition is met.



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Self-similar variables

$$\tilde{X} = \left(\frac{\rho}{\sigma t^2}\right)^{1/3} x \quad ; \quad \tilde{Y} = \left(\frac{\rho}{\sigma t^2}\right)^{1/3} y$$

$$\tilde{\mathbf{X}} := \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \quad ; \quad \tau = \ln(t)$$

Self-similar functions

$$\mathbf{u}(x, y, t) = \left(\frac{\sigma}{\rho t}\right)^{1/3} \tilde{\mathbf{U}}\left[\tilde{X}(x, t), \tilde{Y}(y, t), \tau(t)\right]$$

$$p(x, y, t) = \rho \left(\frac{\sigma}{\rho t}\right)^{2/3} \tilde{P}\left[\tilde{X}(x, t), \tilde{Y}(y, t), \tau(t)\right]$$

Special Notations

- The substitution $\tilde{\nu} = \nu \left(\rho/\sigma\right)^{2/3}$
- The notation simplification $\nabla \equiv \nabla_{(\tilde{X}, \tilde{Y})}$
- The free surface $\tilde{F}$
- The curvature $\tilde{\kappa}$

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Self-similar N-S. Equations

$$\nabla \cdot \tilde{\mathbf{U}} = 0 \qquad \partial_\tau \tilde{\mathbf{U}} + \nabla \cdot \left(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}\right) = -\nabla \tilde{P} + \tilde{\nu}\left(\nabla^2 \tilde{\mathbf{U}}\right) e^{-\tau/3} \underbrace{-\tilde{\mathbf{U}}}_{\text{source term}} + \overbrace{\frac{2}{3} \nabla \cdot \left(\tilde{\mathbf{X}} \otimes \tilde{\mathbf{U}}\right)}^{\text{new advection term}}$$

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Init. Cond. propagated

$$t \to 0 \Rightarrow \tilde{F}\left(\tilde{X}, \tilde{Y}\right) \xrightarrow[\|\tilde{\mathbf{x}}\| \to \infty]{} \tilde{X}\tan(\theta_0) - \tilde{Y}$$

Kinematic Condition

$$\tilde{F}_\tau + \left(\tilde{\mathbf{U}} \cdot \nabla\right)\tilde{F} - \frac{2}{3}\left(\tilde{\mathbf{X}} \cdot \nabla\right)\tilde{F} = 0$$

Far-field Condition

$$\tilde{\mathbf{U}} \xrightarrow[\|\tilde{\mathbf{x}}\| \to \infty]{} \mathbf{0}$$

Dynamic Condition

$$\tilde{P}(\tilde{X}, \tilde{Y}, \tau)\Big|_{\tilde{F}=0} = \tilde{\kappa}$$

III - Self-Similar Navier-Stokes solver with *Basilisk*

Two major additions:

1. A new *advection velocity* $\tilde{\mathbf{\Lambda}} := \tilde{\mathbf{U}} - (2/3)\,\tilde{\mathbf{X}} \neq \tilde{\mathbf{U}}$ the *advected velocity*
2. A *source term* $-\tilde{\mathbf{U}}$ in the RHS of the self-similar formulation of the N-S. equations

Libraries used:

```
centered.h -> bcg.h
two-phase.h -> vof.h
contact.h
tension.h
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⟶ ***Modularity** of Basilisk can be exploited to tackle these changes!*

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```
event advection_term (i++,last)
{
  if (!stokes) {
    [...]
    advection ((scalar *){u}, lambdaf, dt, (scalar *){g});
  }
}
```

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In `vof.h`

We call the face vector `lambdaf[]` instead of `uf[]`
and replace each occurrence of the last one by the first one

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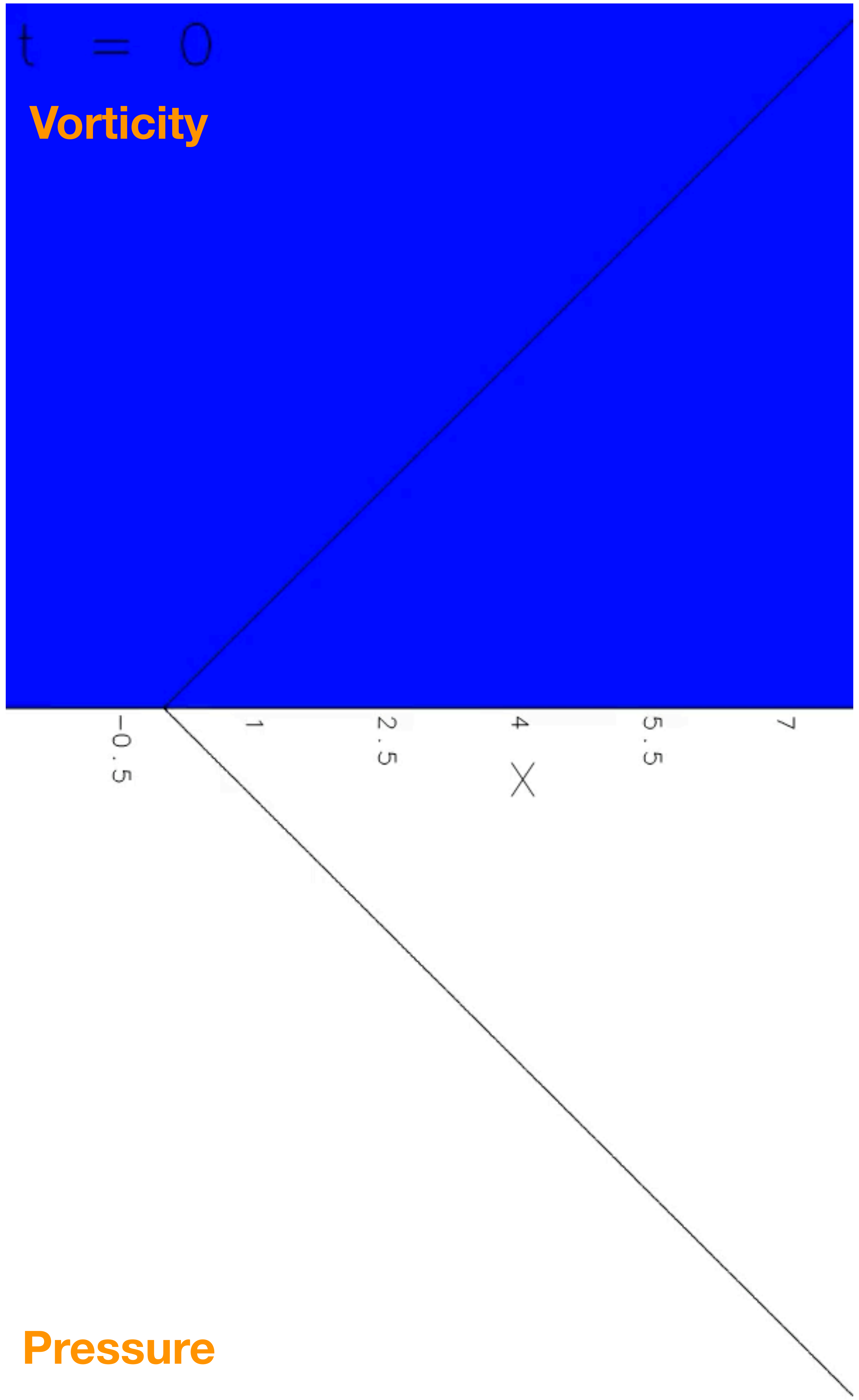
In `user-file.c`

The *source term* $-\tilde{\mathbf{U}}$ in the RHS of the self-similar formulation of the N-S. equations is simply an update of the **`event acceleration`**

$N = 8$
$N_{max} = 9$
$L_0 = 12$
$\theta_0 = 45°$
$\beta_0 = 90°$
$\rho_{fluid} = 1$
$\rho_{gas} = 10^{-3}$
$\mu_{fluid} = 10^{-3}$
$\mu_{gas} = 10^{-5}$
$\sigma = 1$

Global Parameters



III - Self-Sim. N.-S.solver with *Basilisk*

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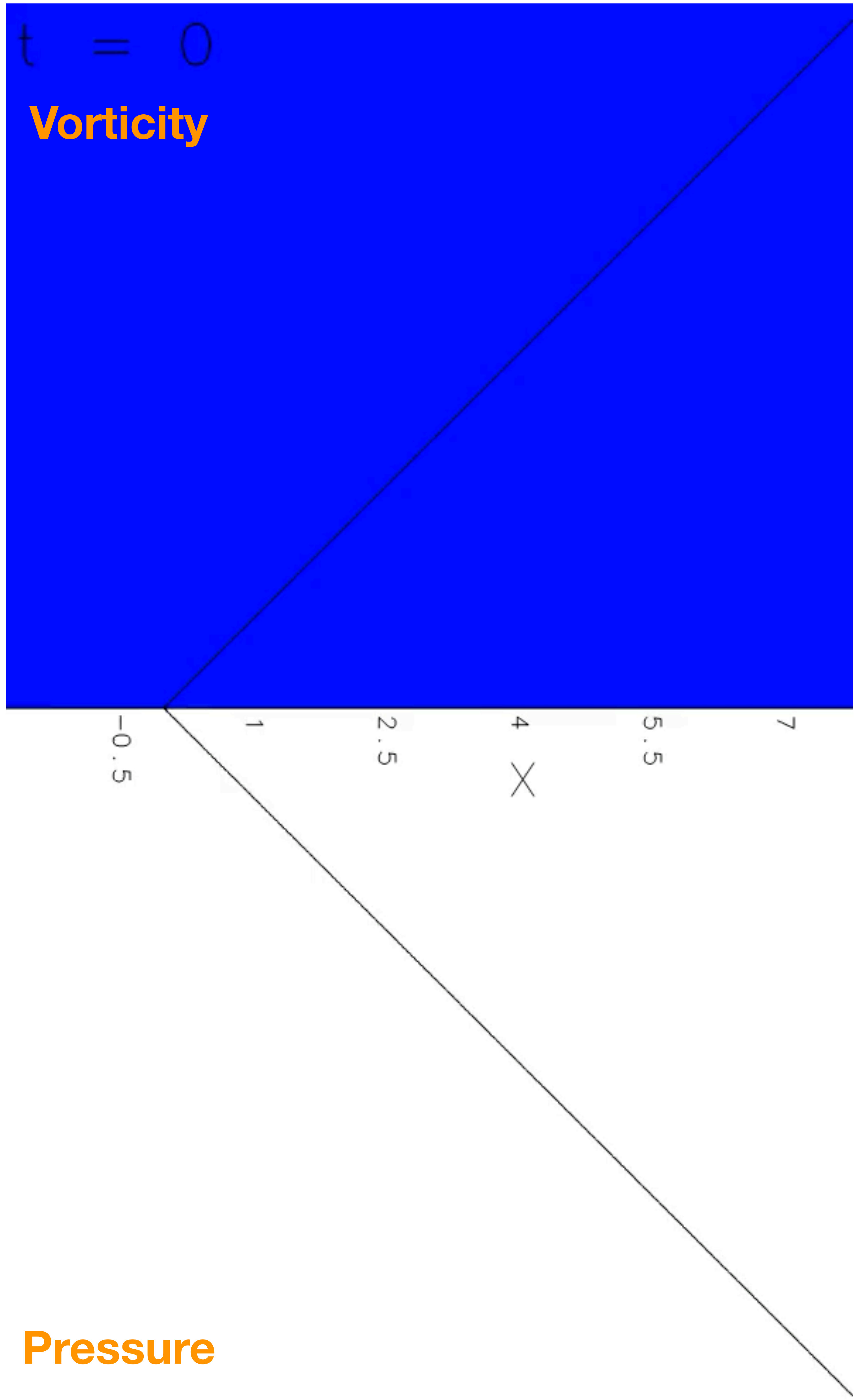

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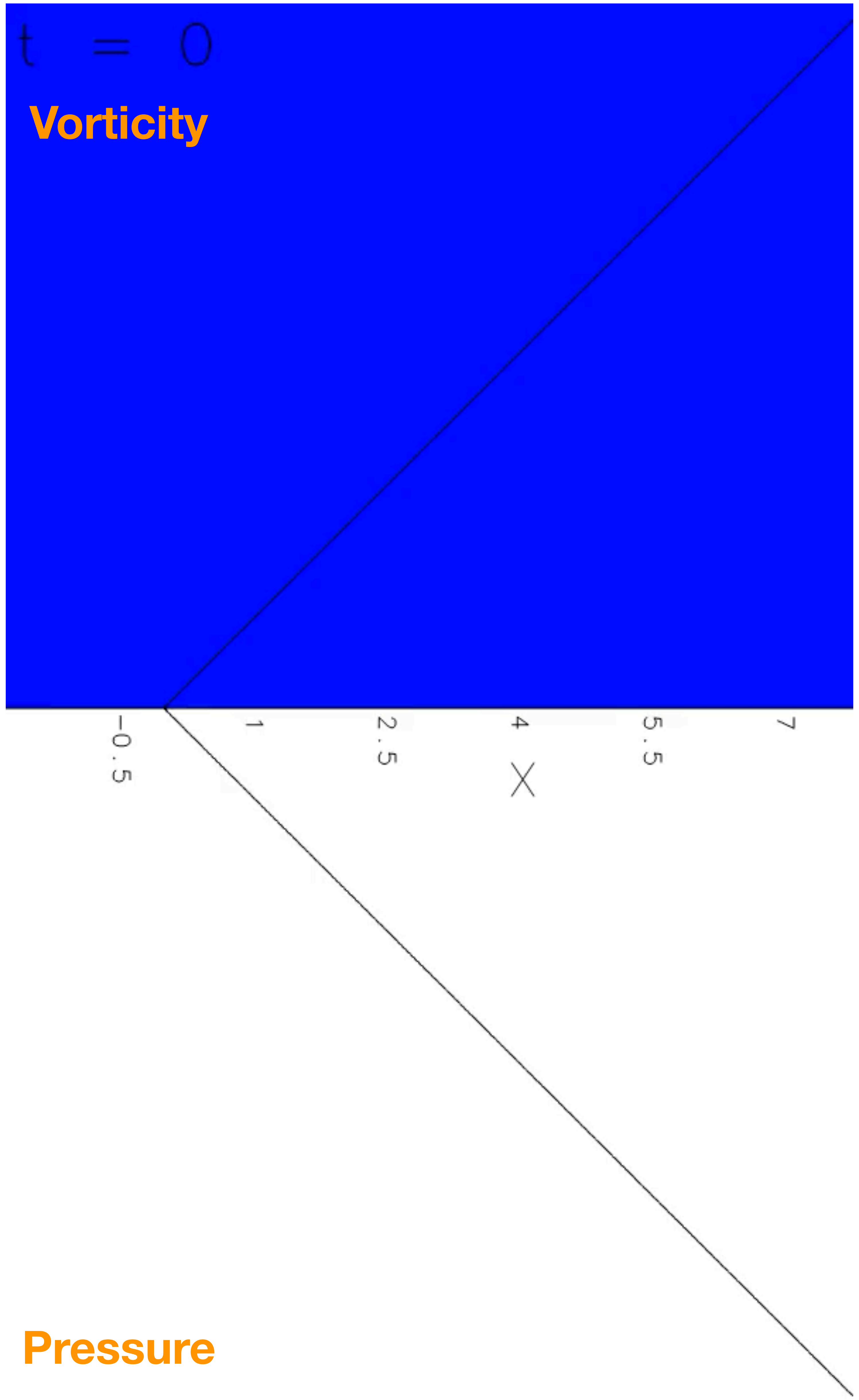

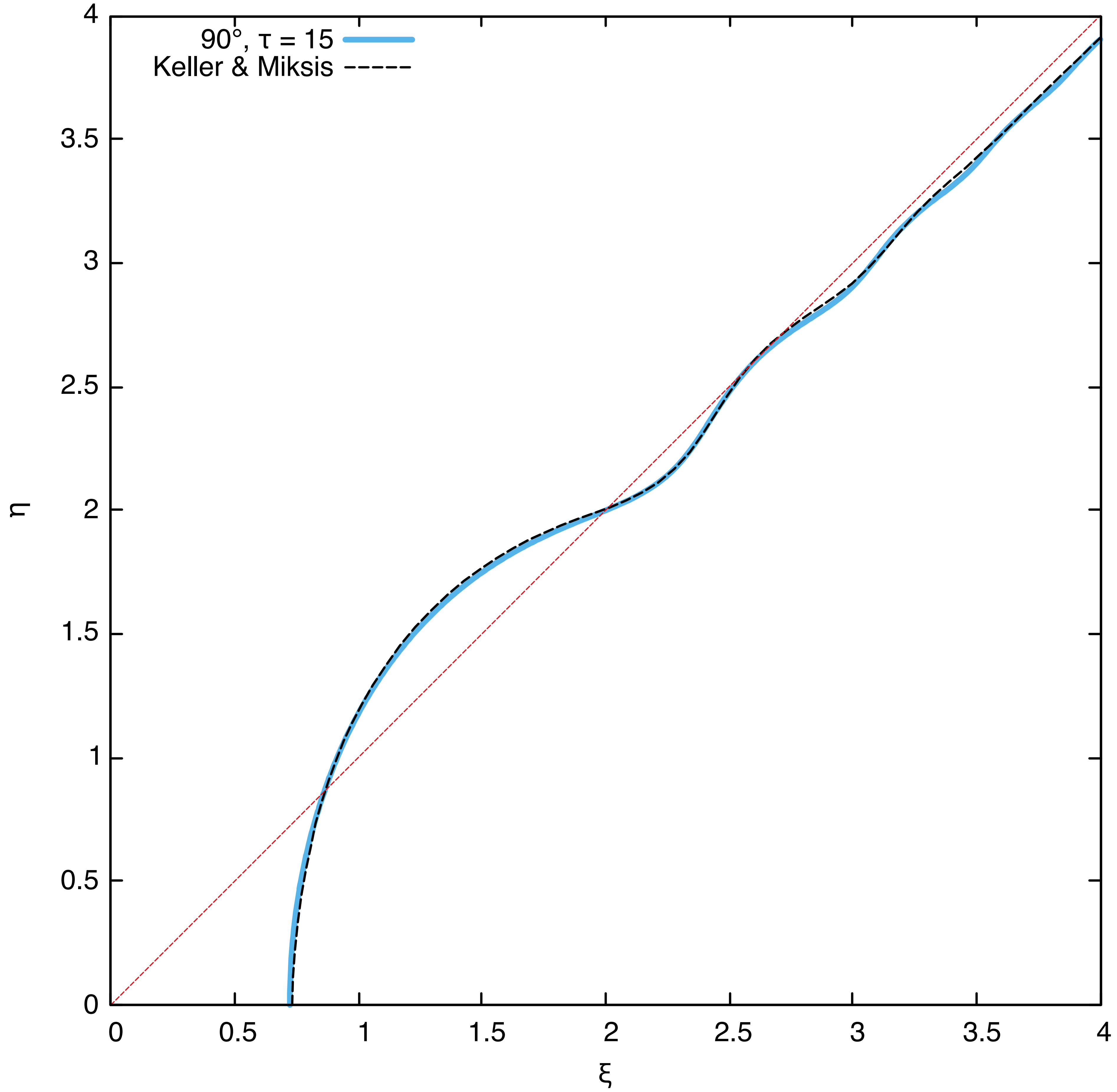

Time-profile showing the achieved convergence towards the self-similar solution

V - Conclusions & Perspectives

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