Higher order methods for Poisson and Advection equations

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Introduction

Motivation for Higher Order Schemes

Order of a scheme (k) : if error $\propto h^k$ (h = grid spacing)

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   - Boundary layer flows

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Advantages of a Higher Order Scheme

1. Higher accuracy for smooth solution
    - Vortex dominated flows (dissipation problem with lower order methods)
    - Boundary layer flows
2. Computation cost
    - For reaching the same error levels, lower grid spacing is required for higher order methods
    - However, complexity of algorithm, means higher CPU time is required for computing each step
    - Tradeoff (Needs to be analyzed)

POISSON PROBLEM - Methodology

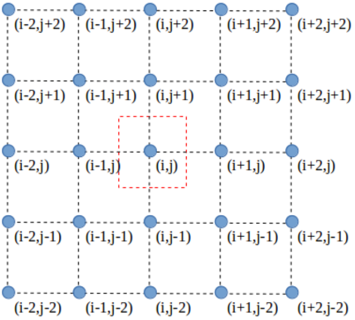
Poisson Helmholtz Equation

The Poisson Helmholtz equation in its generalized formulation

$$L(a) = \nabla \cdot (\alpha \nabla a) + \lambda a = b$$

- Existing Second Order Scheme
- Proposed Fourth Order Scheme

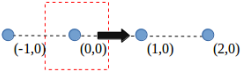
Discretization Scheme - Fourth Order : Stencil Layout



Stencil for building a 4^{th} order scheme

Scheme Development : Needs Computation of face Gradients operator and subsequent Divergence operator !

Discretization Scheme : Face Centred Gradient Operator



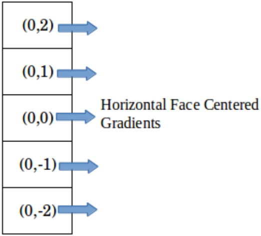
1. X equation :

$$\nabla_x A_{(0.5,0)} = \frac{A(-1,0) - 27A(0,0) + 27A(1,0) - A(2,0)}{24\delta}$$

2. Y equation :

$$\nabla_y A_{(0,0.5)} = \frac{A(0,-1) - 27A(0,0) + 27A(0,1) - A(0,2)}{24\delta}$$

Discretization Scheme : Face Averaged Gradient Operator



- X : $(\nabla_x)^{avg} A_{(0.5,0)} = \frac{-17\nabla_x A_{(0.5,2)} + 308\nabla_x A_{(0.5,1)} + 5178\nabla_x A_{(0.5,0)} + 308\nabla_x A_{(0.5,-1)} - 17\nabla_x A_{(0.5,-2)}}{5760}$
- Y : $(\nabla_y)^{avg} A_{(0,0.5)} = \frac{-17\nabla_y A_{(2,0.5)} + 308\nabla_y A_{(1,0.5)} + 5178\nabla_y A_{(0,0.5)} + 308\nabla_y A_{(-1,0.5)} - 17\nabla_y A_{(-2,0.5)}}{5760}$

- $\nabla^2 A_{(0,0)} = \frac{(\nabla_x)^{avg} A_{(0.5,0)} - (\nabla_x)^{avg} A_{(-0.5,0)} + (\nabla_y)^{avg} A_{(0,0.5)} - (\nabla_y)^{avg} A_{(0,-0.5)}}{\delta}$

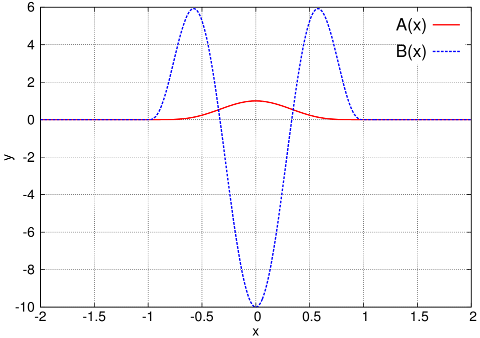
Poisson solver algorithm

1. Jacobi Iterations
2. Multigrid method
    - Restriction of Residuals from finest cells down to coarsest cells
    - Prolongation of Iterative solution corrections from coarsest to finest levels
    - Accelerates the algorithm. Large wavelength errors die out faster on coarser grids
3. Adaptivity

Test Function - Compact Support Function

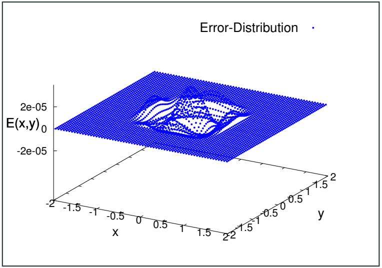
$$A(x) = \begin{cases} (1 - x^2)^5 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

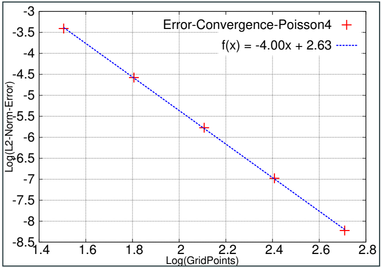
$$B(x) = \nabla^2 A(x)$$



Results

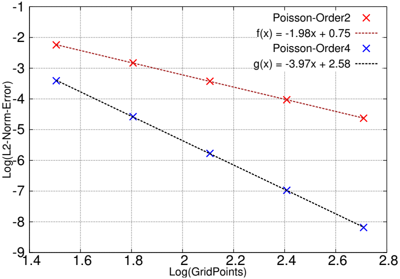
Uniform grid





The scheme shows an error convergence of 4 as expected !

Order 2 vs Order 4 simulation



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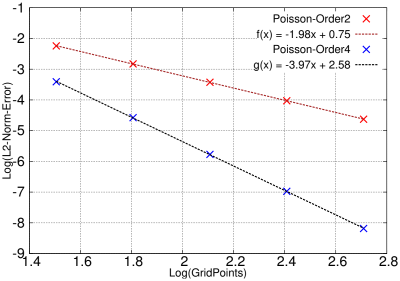

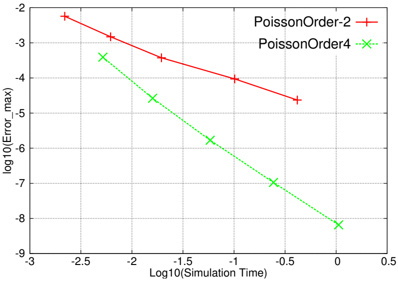

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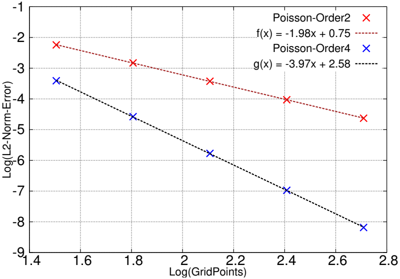

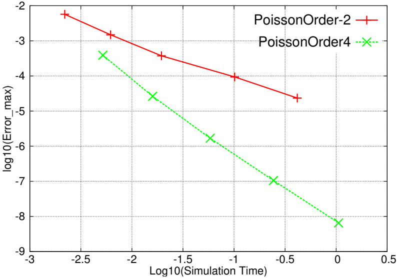

- To achieve an L1 error of 1E-04 in the domain :
    1. Poisson order 2 scheme requires 256*256 grid points. Simulation time ∼ 0.101 seconds.
    2. Poisson order 4 scheme requires 45*45 grid points. Simulation time ∼ 0.011 seconds.

Motivation for WENO Schemes

2D Saint Venant Equations

1. **Assumptions**
   - Horizontal length scale >> Vertical length scale
   - Vertical velocity << Horizontal Velocity
   - Pressure variation in vertical direction is a hydrostatic variation

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   - Horizontal length scale >> Vertical length scale
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2. **SV equation derivation**
   - Obtained by depth Integrating the Navier Stokes equations

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x} + \frac{\partial (huv)}{\partial y} = 0$$

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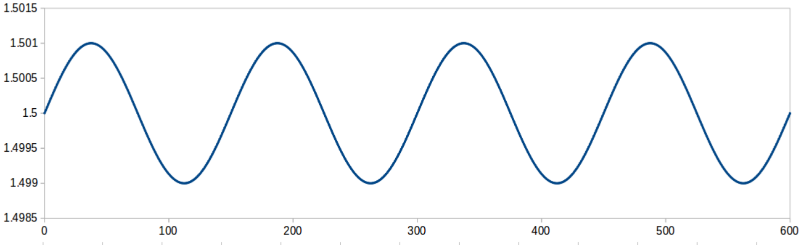
1D Gravity Wave test case



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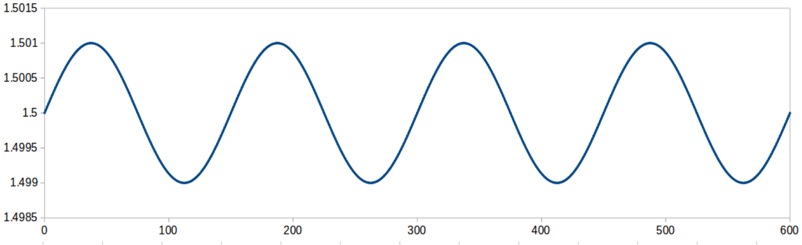
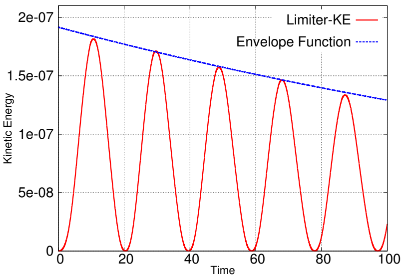


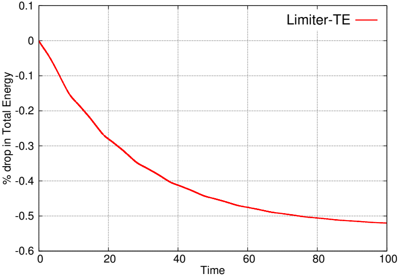
- $\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$
- $\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2} g h^2)}{\partial x} = 0$
- Periodic boundary conditions
- Probe point : Last crest point

1D Gravity Wave - Flux Limiter Results

Numerical Algorithm

- Predictor-corrector algorithm for time advection
- Minmod 2 Limiter used for face flux computation
- Riemann problem - Kurganov Method





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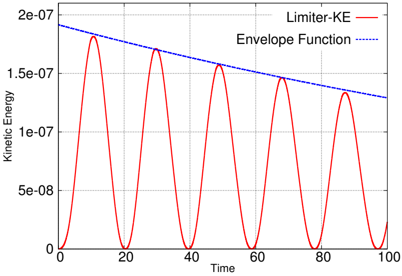
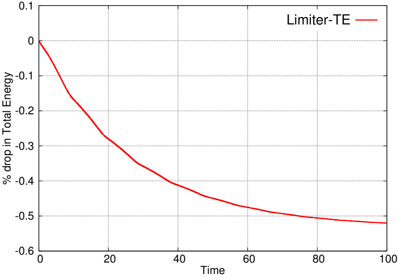
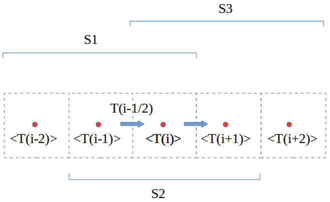




Motivation : To build an advection scheme, which cuts down on numerical damping !

WENO Methodology

WENO Schemes 1D - Methodology

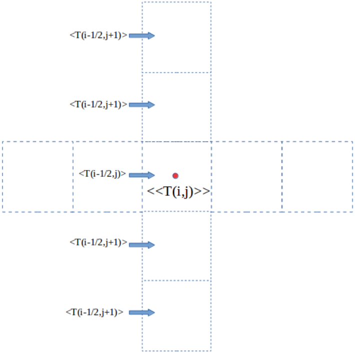


- 3rd Order polynomial interpolations of volume averages to face values (S1, S2 and S3)
- Limiting function derivations in each stencil (Measures smoothness of Interpolant)

$$\beta_j = \sum_{l=1}^{k} \Delta x^{2l-1} \int_{x_{i-1/2}}^{x_{i+1/2}} \left(\frac{d^l}{dx^l} P_j(x)\right)^2 dx$$

- Derive weight functions using the limiters (Convex combination of all three stencil contributions)
- Order 5 interpolation achieved in smooth regions, while order 3 in discontinuous regions.

WENO Schemes 2D - Methodology



- 1D Weno Sweep. Compute face avg values using Surface avg values.

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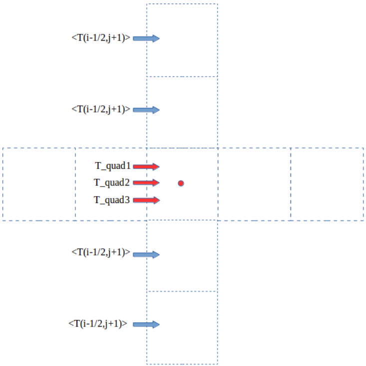
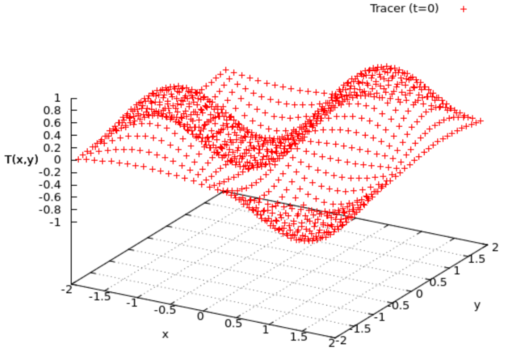


- 1D Weno sweep. Compute face avg values using Surface avg values.
- 1D Weno sweep in transverse direction. Compute point values at 3 quadrature points.
- Solve the riemann problem to Compute point fluxes at all three quadrature points.
- Guassian quadrature sum to get flux over entire face.

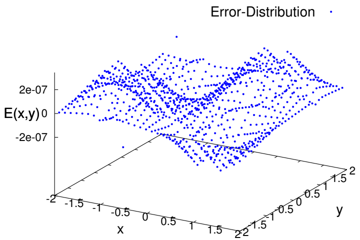
Demonstrating Weno Advection (Test case setup)

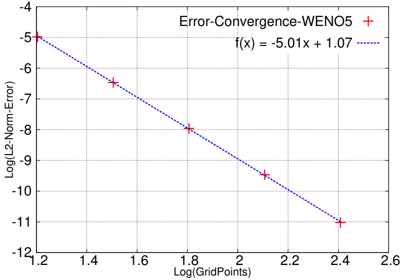
- Tracer Initialization with a periodic function $Tracer(t = 0) = sin(\pi x/2)sin(\pi y/2)$
- Constant advecting velocity $u = 3$, $v = 2$
- Periodic boundary conditions



Error Distribution and Convergence

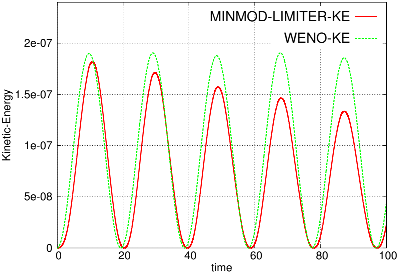
- Uniform Grid 16*16, 32*32, 64*64, 128*128 and 256*256
- WENO 5 advection schemes
- Runge Kutta 4 time step marching
- Demonstrates Order 5 error convergence





Results - Gravity Wave

Minmod Limiter results vs Weno schemes



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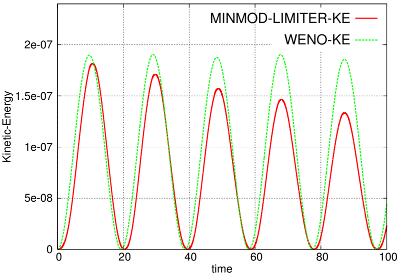

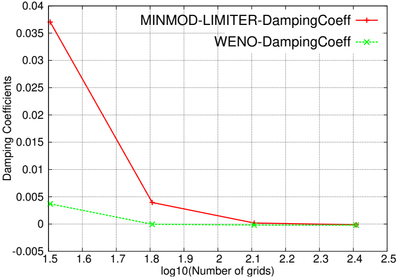

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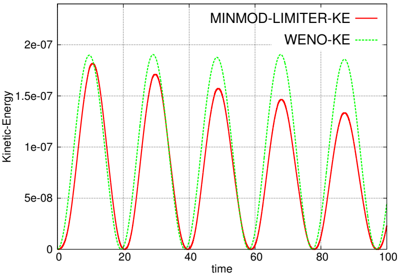

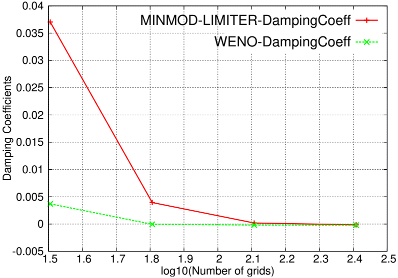

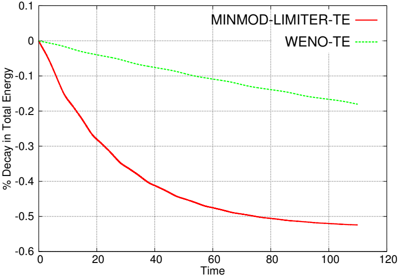

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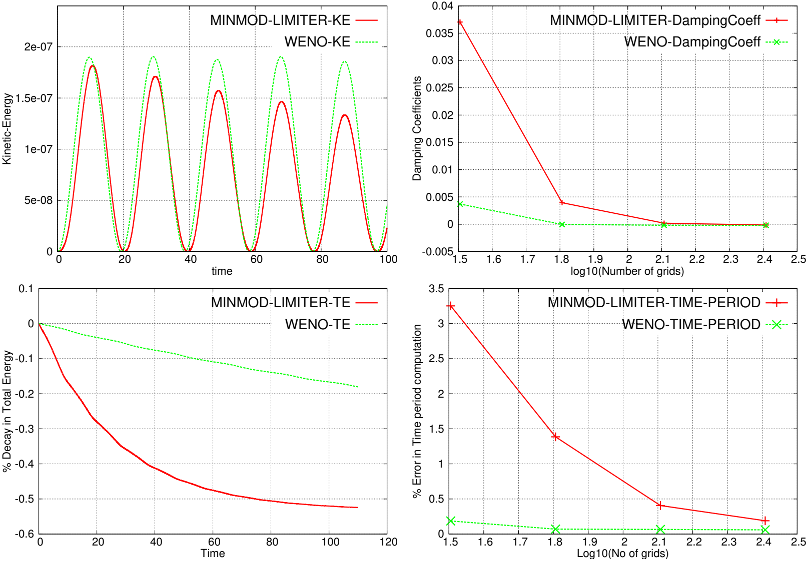

Adaptivity Methodology

Adaptive Grids in Basilisk

- Basilisk supports tree based grids (convenient for refinement / coarsening action)
- Basilisk employs an adaptive wavelet algorithm (which is not problem specific)

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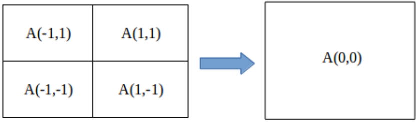
  - The error is given by the difference of the original node cell values with the Restricted-Prolongated values
  - The cells which show an error higher than the upper-tolerance limit are refined
  - The cells which show an error lower than the lower-tolerance limit are coarsened

Restriction operator



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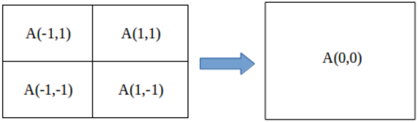
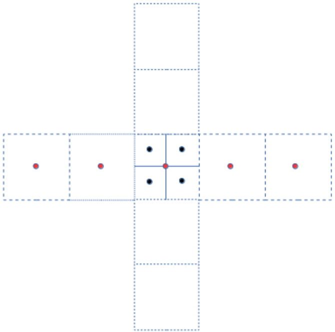


The restriction operator is quite basic. It is the direct average over the four children cells.

$$A(0,0) = \frac{A(-1,-1) + A(-1,1) + A(1,-1) + A(1,1)}{4}$$

Prolongation Operator



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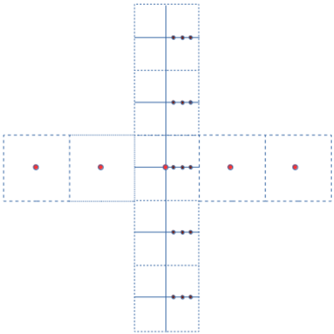

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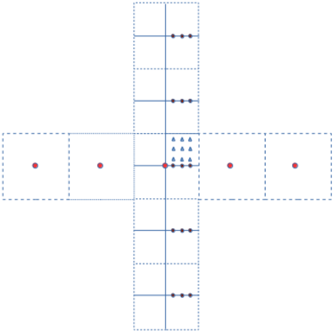

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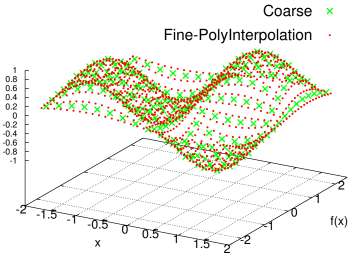
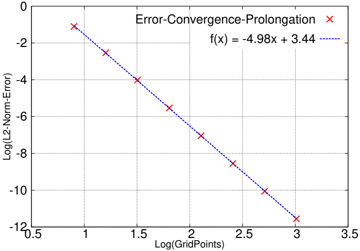
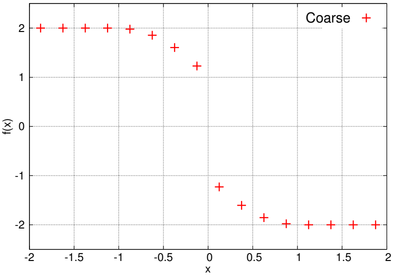
Prolongation operator - Smooth Functions

- $f(x, y) = sin(\pi x/2) sin(\pi y/2)$
- Demonstrates Order 5 error convergence





Prolongation operator - Discontinuous functions



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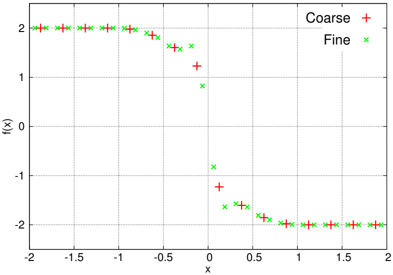

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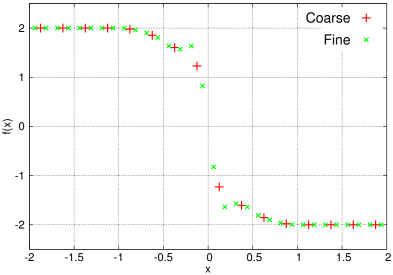

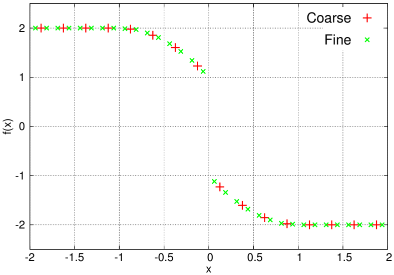

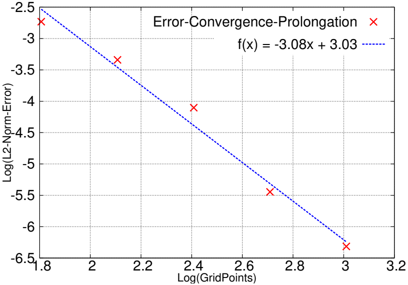

Conclusion

Remarks on higher order schemes

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2. The Weno 5 advection schemes have a superior performance to the minmod 2 limiter scheme, when it comes to analyzing the numerical dispersion and the dissipation of the schemes.

3. Higher order error estimation for adaptive mesh reconstruction has been succesfully demonstrated.

Thank You !