Bingham & Granular flows with *Gerris* and *Basilisk*

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Basilisk/Gerris Users' Meeting 2017 Nov 16th

http://basilisk.fr/BGUM2017

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Jean Le Rond ∂'Alembert
1717 1783

- Bingham rheology
- Granular μ(I) rheology
- Implementation in *Basilisk*

Granulars

- Example of column collapse
- Examples of silo

Bingham

- Hierarchy of models for Bingham

Jean Le Rond ∂'Alembert
1717 1783

motivation

facts:

sand, granulates: 6 10^3 kg/french/year

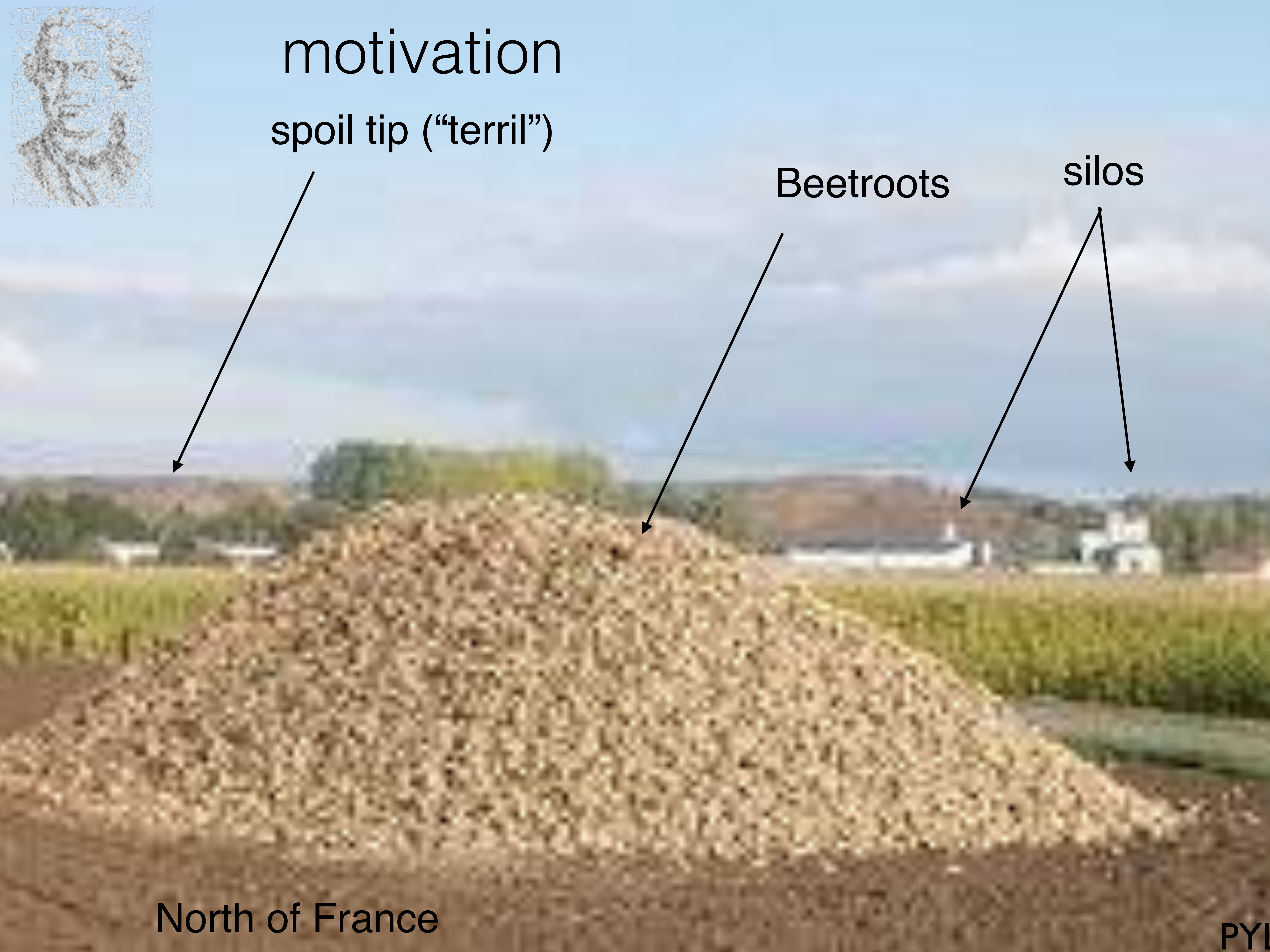
http://www.brgm.fr/sites/default/files/btp_approvisionnement_materiaux_0.pdf

second most used "fluid" water (1./10*)
far ahead petroleum (4*)

https://www.planetoscope.com/petrole/1480-consommation-de-petrole-en-france.html

Food, medicine

Environmental flows (avalanches, mud flow….)


motivation
spoil tip ("terril")
Beetroots
silos
North of France

motivation

Rio de Janeiro, Ipanema

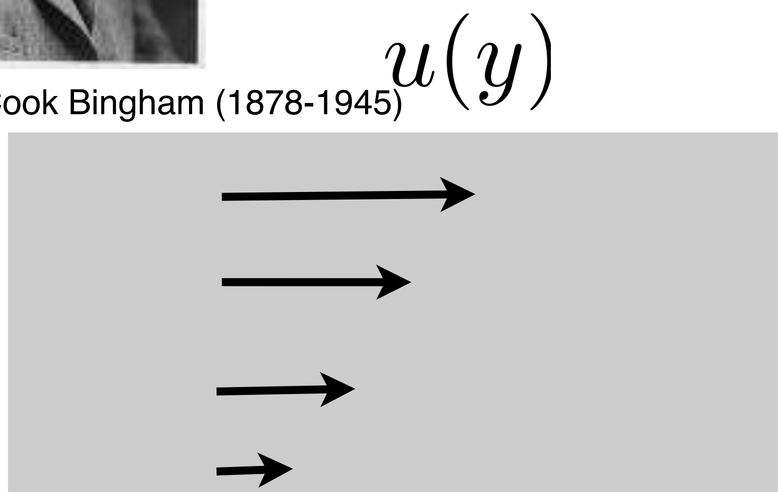
PYL

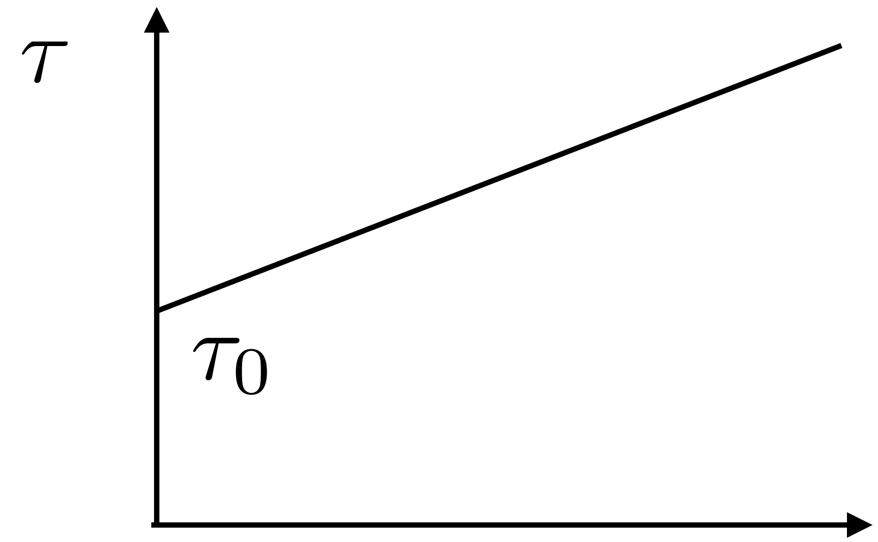
motivation

Paris, Univ. Pierre & Marie Curie, University Paris 06, Jussieu

The Bingham-rheology

Eugene Cook Bingham (1878-1945)





$$\dot{\gamma} = \frac{\partial u}{\partial y}$$

yield stress — viscosity

$$\tau = \tau_0 + \eta_0 \frac{\partial u}{\partial y}$$

«Bingham Number» $Bi = \frac{\tau_0}{\eta_0 U/L}$

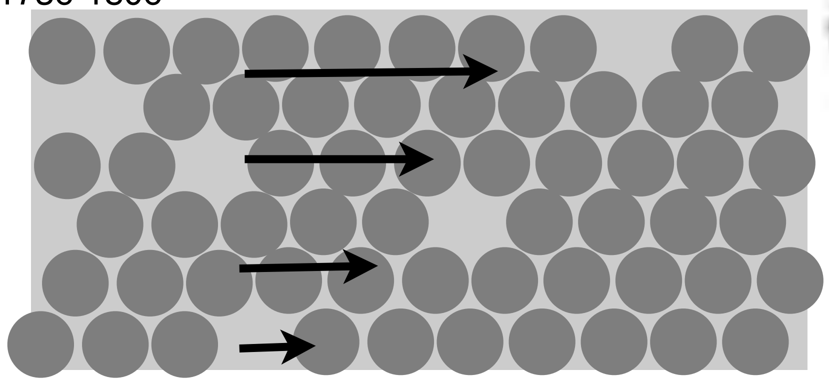
$$\frac{\text{yield}}{\text{viscosity}}$$

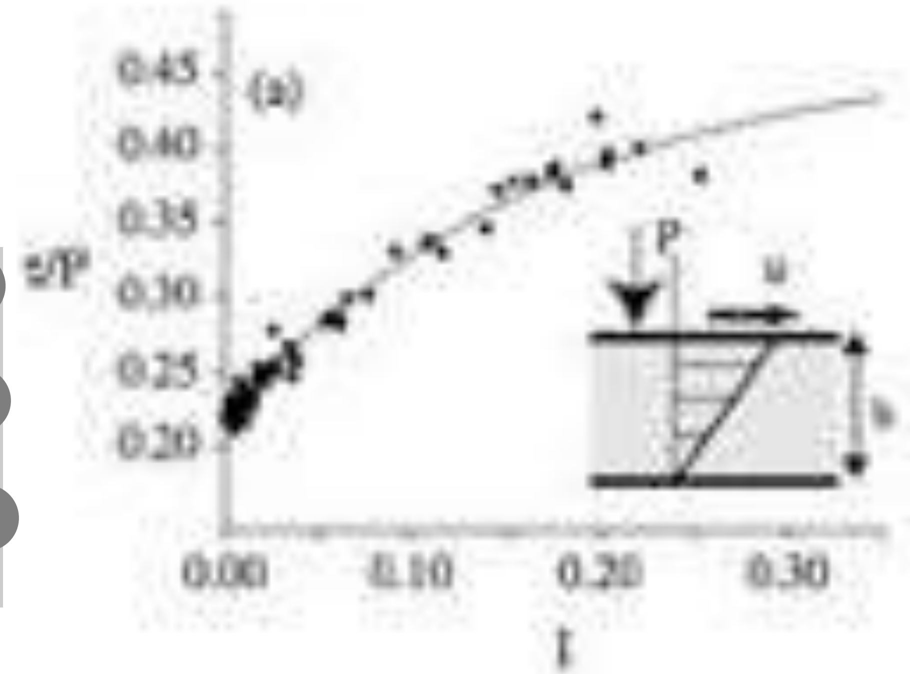
The μ(I)-rheology

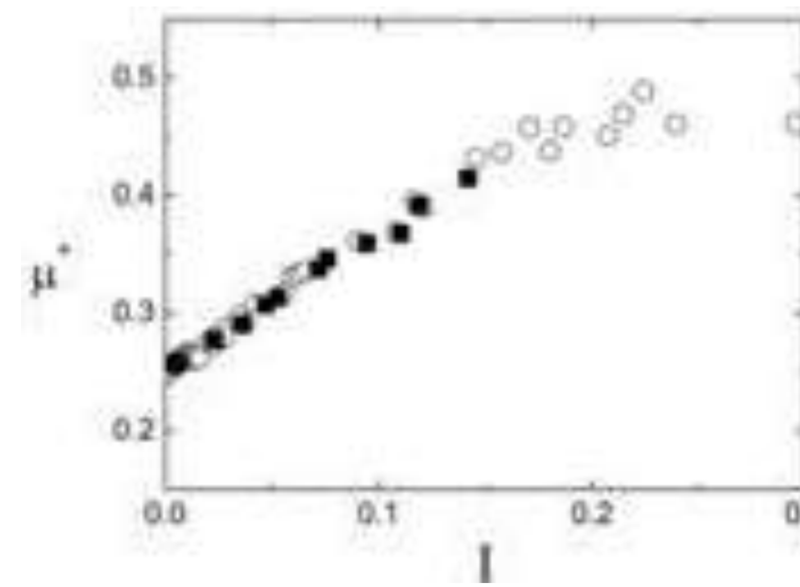
by grain dynamics

Charles-Augustin Coulomb
1736-1806

$u(y)$







Da Cruz PRE 05

Coulomb friction law

«Drucker-Prager» plastic flow

$$\tau = \mu(I) P$$

«Inertial Number»

$$I = \frac{d\frac{\partial u}{\partial y}}{\sqrt{P/\rho}}$$

$$\frac{\textbf{falling time}}{\textbf{displacement time}}$$

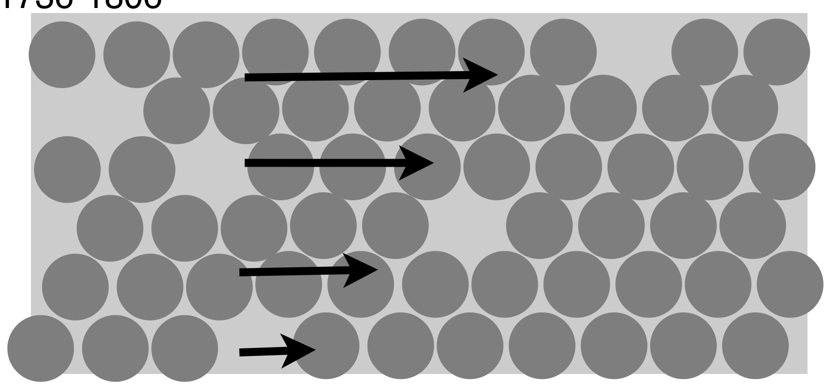
non dimensional number: «Froude»
local «Inertial Number» (Da Cruz 04-05)
(Savage Number 89 / Ancey 00 I^2)

Pouliquen 99
Pouliquen Forterre JSM 06
Da Cruz 04-05
GDR Midi 04

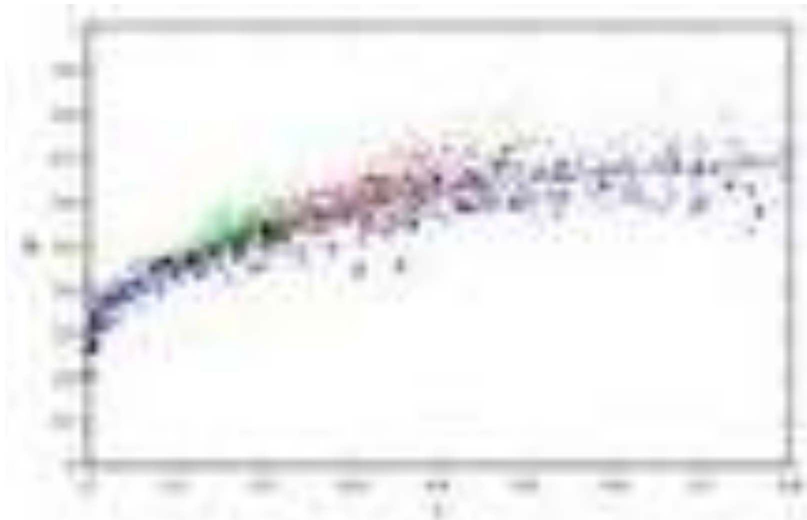
Charles-Augustin Coulomb
1736-1806

The μ(I)-rheology

$u(y)$



by grain dynamics



Lacaze Kerswell 09

Coulomb friction law

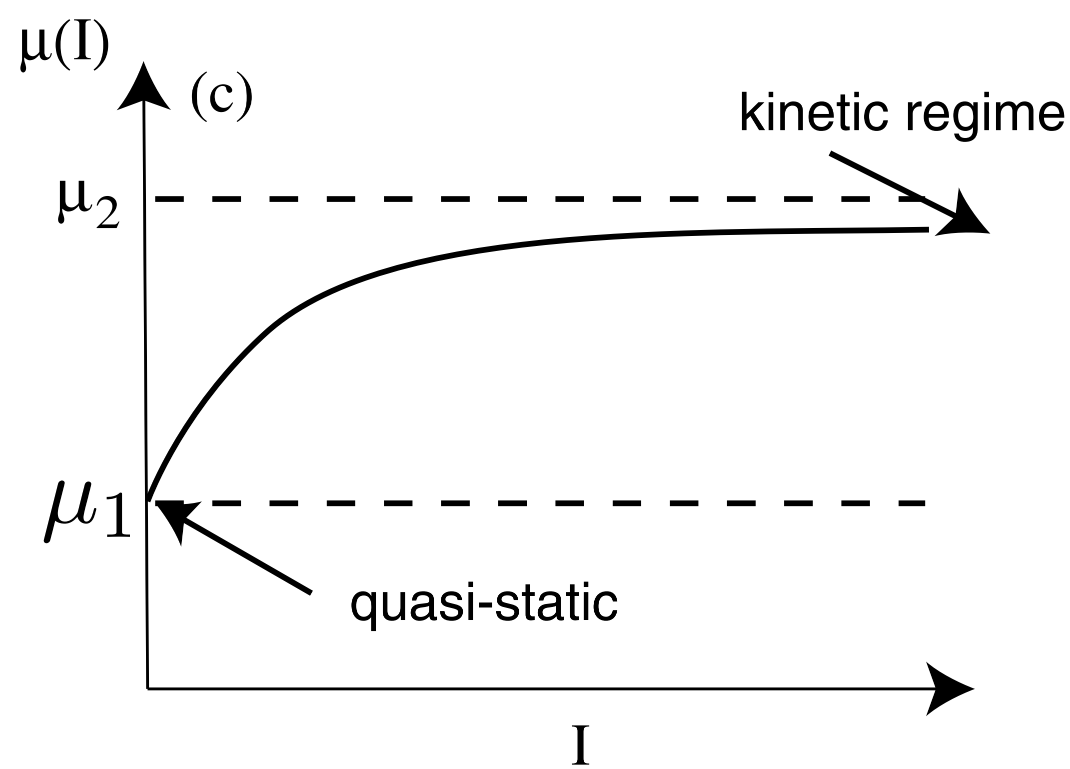
$$\tau = \mu(I) P$$

«Inertial Number»

$$I = \frac{d \frac{\partial u}{\partial y}}{\sqrt{P/\rho}}$$

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1}$$

$$\mu_1 \simeq 0.32 \quad (\mu_2 - \mu_1) \simeq 0.23 \quad I_0 \simeq 0.3$$



Jop Forterre Pouliquen 2005

General formulation

Non newtonian flows:
local constitutive law

(Stokesian or Reiner Rivlin)

first order fluids
(linear Stokesian or linear Reiner Rivlin)

simple form $\dot{\gamma} = \frac{\partial u}{\partial y}$

$$\tau = \left(\eta(\frac{\partial u}{\partial y})\right) \frac{\partial u}{\partial y}$$

$$\sigma_{ij} = f(D_{ij})$$

With strain rate tensor

$$D_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

second invariant of strain rate tensor

$$D_2 = \sqrt{D_{ij} D_{ij}}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\eta(D_2) D_{ij}$$

tensorial formulation

$$\tau_{ij} = 2(\eta(D_2)) D_{ij}$$

Example Bingham

classic formulation

$$\tau = \tau_0 + \eta_0 \frac{\partial u}{\partial y}$$

practical formulation

$$\tau = \left(\frac{\tau_0}{\frac{\partial u}{\partial y}} + \eta_0 \right) \frac{\partial u}{\partial y}$$

$$\tau = \left(\eta(\frac{\partial u}{\partial y}) \right) \frac{\partial u}{\partial y}$$

$$\sigma_{ij} = f(D_{ij})$$

With strain rate tensor

$$D_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

second invariant of strain rate tensor

$$D_2 = \sqrt{D_{ij} D_{ij}}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\eta(D_2) D_{ij}$$

tensorial formulation

$$\tau_{ij} = 2(\eta(D_2)) D_{ij}$$

$$\eta = (\eta_0 + \frac{\tau_0}{\sqrt{2} D_2})$$

Example Granular μ(I) flows

classic formulation

$$\tau = \mu p$$

practical formulation

$$\tau = \left(\frac{\mu p}{\frac{\partial u}{\partial y}} \right) \frac{\partial u}{\partial y}$$

$$\tau = \left(\eta(\frac{\partial u}{\partial y}) \right) \frac{\partial u}{\partial y}$$

$$\sigma_{ij} = f(D_{ij})$$

With strain rate tensor

$$D_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

second invariant of strain rate tensor

$$D_2 = \sqrt{D_{ij} D_{ij}}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\eta(D_2) D_{ij}$$

tensorial formulation

$$\tau_{ij} = 2(\eta(D_2)) D_{ij}$$

$$\eta = \left(\frac{\mu(I)}{\sqrt{2} D_2} p \right)$$

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1}$$

$$I = d\sqrt{2} D_2 / \sqrt{(|p|/\rho)}.$$

implementation in *Gerris/ Basilisk* flow solver?

$$\sigma_{ij} = -p\delta_{ij} + 2\eta(D_2)D_{ij}$$

$$D_2 = \sqrt{D_{ij}D_{ij}} \qquad D_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

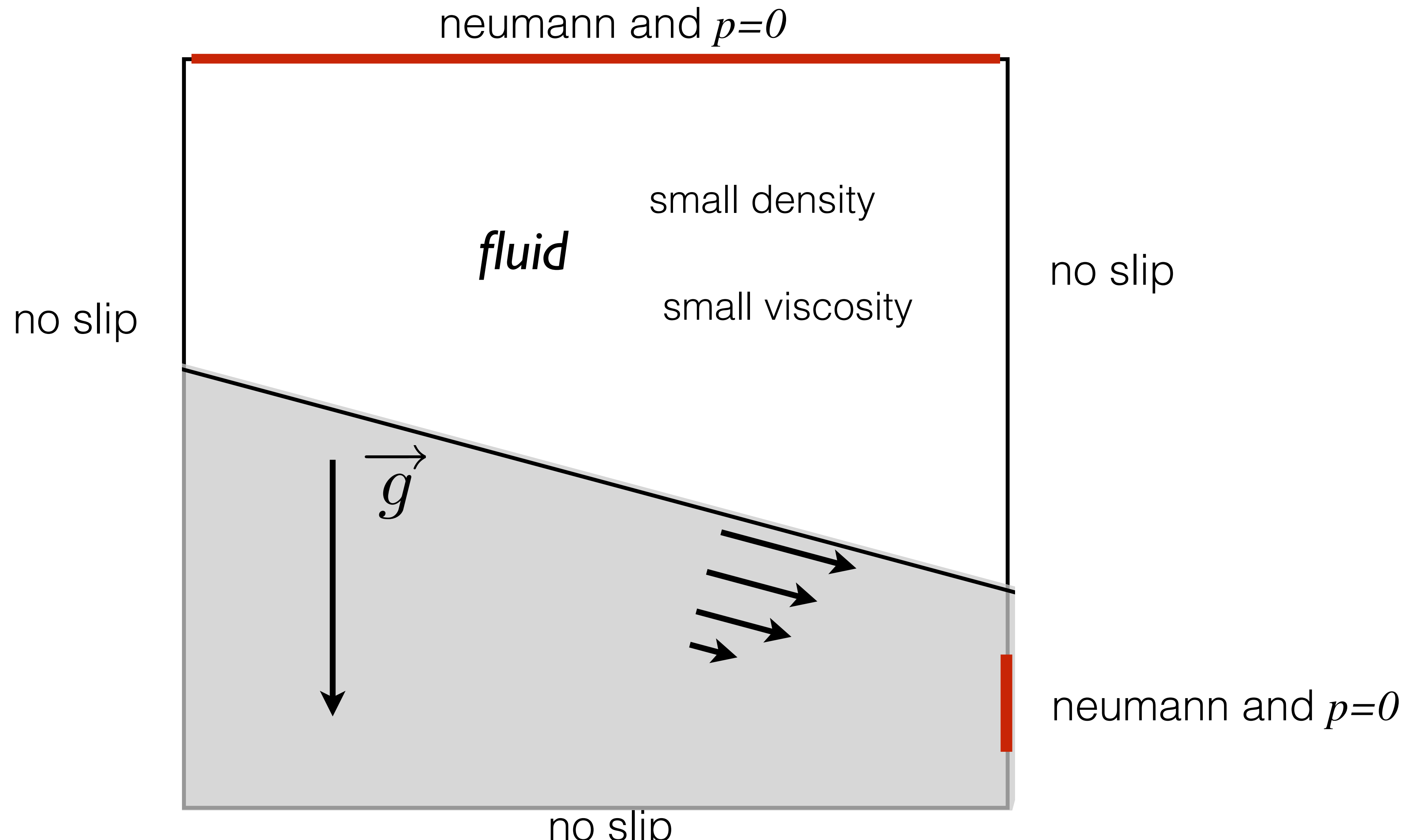
construction of a viscosity based on the D_2 invariant and redefinition of I

$$\eta = \min(\eta_{max}, \max\left(\eta(D_2)\ ,\ 0\right)) \qquad I = d\sqrt{2}D_2/\sqrt{(|p|/\rho)}.$$

- the «min» limits viscosity to a large value
- always flow, even slow

Boundary Conditions: no slip and $p=0$ at the interface for μ(I)

implementation in *Gerris/ Basilisk* flow solver?



$$\nabla \cdot \mathbf{u} = 0, \quad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nabla \cdot (2\eta \mathbf{D}) + \rho g,$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{u}) = 0, \quad \rho = c\rho_1 + (1-c)\rho_2, \quad \eta = c\eta_1 + (1-c)\eta_2$$

The granular fluid is covered by a passive light fluid (it allows for a zero pressure boundary condition at the surface, bypassing an up to now difficulty which was to impose this condition on a unknown moving boundary).

Boundary Conditions: no slip and $p=0$ at the interface for μ(I)

implementation in *Gerris/ Basilisk* flow solver?

Projection Method

$$\rho_{n+\frac{1}{2}}\left(\frac{\mathbf{u}_*-\mathbf{u}_n}{\Delta t}+\mathbf{u}_{n+\frac{1}{2}}\cdot\boldsymbol{\nabla}\mathbf{u}_{n+\frac{1}{2}}\right)=\boldsymbol{\nabla}\cdot(\eta_{n+\frac{1}{2}}\mathbf{D}_*)-\boldsymbol{\nabla}p_{n-\frac{1}{2}},$$

$$\mathbf{u}_{n+1}=\mathbf{u}_*-\frac{\Delta t}{\rho_{n+\frac{1}{2}}}(\boldsymbol{\nabla}p_{n+\frac{1}{2}}-\boldsymbol{\nabla}p_{n-\frac{1}{2}}),$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}_{n+1}=0.$$

multigrid solver for Laplacian of pressure

$$\boldsymbol{\nabla}\cdot\left(\frac{\Delta t}{\rho_{n+\frac{1}{2}}}\boldsymbol{\nabla}p_{n+\frac{1}{2}}\right)=\boldsymbol{\nabla}\cdot\left(\mathbf{u}_*+\frac{\Delta t}{\rho_{n+\frac{1}{2}}}\boldsymbol{\nabla}p_{n-\frac{1}{2}}\right)$$

implicit for $\mathbf{u}^\star$

$$\frac{\rho_{n+\frac{1}{2}}}{\Delta t}\mathbf{u}_\star-\frac{1}{2}\nabla\cdot(\eta_{n+\frac{1}{2}}\nabla\mathbf{u}_\star)\quad=\quad\rho_{n+\frac{1}{2}}\left[\frac{\mathbf{u}_n}{\Delta t}-\mathbf{u}_{n+\frac{1}{2}}\cdot\nabla\mathbf{u}_{n+\frac{1}{2}}\right]-\nabla p_{n-\frac{1}{2}}+\frac{1}{2}\nabla\mathbf{u}_n^T\nabla\eta_{n+\frac{1}{2}}.$$

VOF reconstruction

$$\frac{c_{n+\frac{1}{2}}-c_{n-\frac{1}{2}}}{\Delta t}+\nabla\cdot(c_n\mathbf{u}_n)=0$$

implementation Bingham

$$D_2 = \sqrt{D_{ij}D_{ij}}$$

$$\eta = \eta_0 + \frac{\tau_0}{\sqrt{2}D_2}$$

$$Bi = \tau_0/(\eta_0 U/L)$$

```
#include "navier-stokes/centered.h"
```

```
event properties (i++) {
  trash ({alphav});
  scalar η[];
  foreach() {

  … compute η

  }
  boundary ({η});

  scalar fa[];
  foreach()
    fa[] = (4.*f[] +
      2.*(f[-1,0] + f[1,0] + f[0,-1] + f[0,1]) +
      f[1,1] + f[-1,1] + f[1,-1] + f[-1,-1])/16.;
  boundary ({fa});

  foreach_face() {
    double fm = (fa[] + fa[-1,0])/2.;
    muv.x[] = (fm*(η[] + η[-1,0])/2. + (1. - fm)*mug);
    alphav.x[] = 1./ρ(fm);
  }
foreach()
    rhov[] = ρ(fa[]);
 boundary ({muv,alphav,rhov});
}
```

```
foreach() {
    η[] = etamx;
    double D2 = 0.;
    foreach_dimension() {
        double dxx = u.x[1,0] - u.x[-1,0];
        double dxy = (u.x[0,1] - u.x[0,-1] + u.y[1,0] - u.y[-1,0])/2.;
        D2 += sq(dxx) + sq(dxy);
    }
    if (D2 > 0.) {
        D2 = sqrt(D2)/(2.*Δ);
        η[] = (Bi/(sqrt(2)*D2 + eta0*Bi/etamx) + 1)*eta0 ;
    }
}
```

implementation μ(I)

$$D_2 = \sqrt{D_{ij}D_{ij}}$$

$$I = d\sqrt{2}D_2/\sqrt{(|p|/\rho)}.$$

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{I_0/I + 1}$$

$$\eta = \left(\frac{\mu(I)}{\sqrt{2}D_2}p\right)$$

```
#include "navier-stokes/centered.h"
```

```
event properties (i++) {
  trash ({alphav});
  scalar η[];
  foreach() {

  … compute η

  }
  boundary ({η});

  scalar fa[];
  foreach()
    fa[] = (4.*f[] +
      2.*(f[-1,0] + f[1,0] + f[0,-1] + f[0,1]) +
      f[1,1] + f[-1,1] + f[1,-1] + f[-1,-1])/16.;
  boundary ({fa});

  foreach_face() {
    double fm = (fa[] + fa[-1,0])/2.;
    muv.x[] = (fm*(η[] + η[-1,0])/2. + (1. - fm)*mug);
    alphav.x[] = 1./ρ(fm);
  }
foreach()
    rhov[] = ρ(fa[]);
 boundary ({muv,alphav,rhov});
}
```

```
foreach() {
  η[] = mug;
  if (p[] > 0.) {
    double D2 = 0.;
    foreach_dimension() {
double dxx = u.x[1,0] - u.x[-1,0];
double dxy = (u.x[0,1] - u.x[0,-1] + u.y[1,0] - u.y[-1,0])/2.;
D2 += sq(dxx) + sq(dxy);
    }
    if (D2 > 0.) {
D2 = sqrt(D2)/(2.*Δ);  // this is D2

double sD2 = sqrt(2.)*D2; // this sD2 is (sqrt(2) D2)
double In = sD2*Dgrain/sqrt(p[]);
double muI = .4 + .28*In/(.4 + In);
double etamin = sqrt(Dgrain*Dgrain*Dgrain);
η[] = max((muI*p[])/sD2, etamin);
η[] = min(η[],100);      }
  }
}
boundary ({η});
scalar fa[];
```

Implementations of NS-μ(I)

- Mangeney, Ionescu, Bouchut, Lusso 2016
- Krabbenhoft 2014
- Dunatunga & Kamrin 2015
- Barker & Gray 2015
- Daviet & Bertails-Descoubes 2016

Implementations of Bingham

- Liu, Balmforth, Hormozi, Hewitt, 2016,
- Dufour and Pijaudier-Cabotz 2005
- Vinay Wachs, Agassant 2005
- Vola, Babik, Latché 2004

Jean Le Rond ∂'Alembert
1717 1783

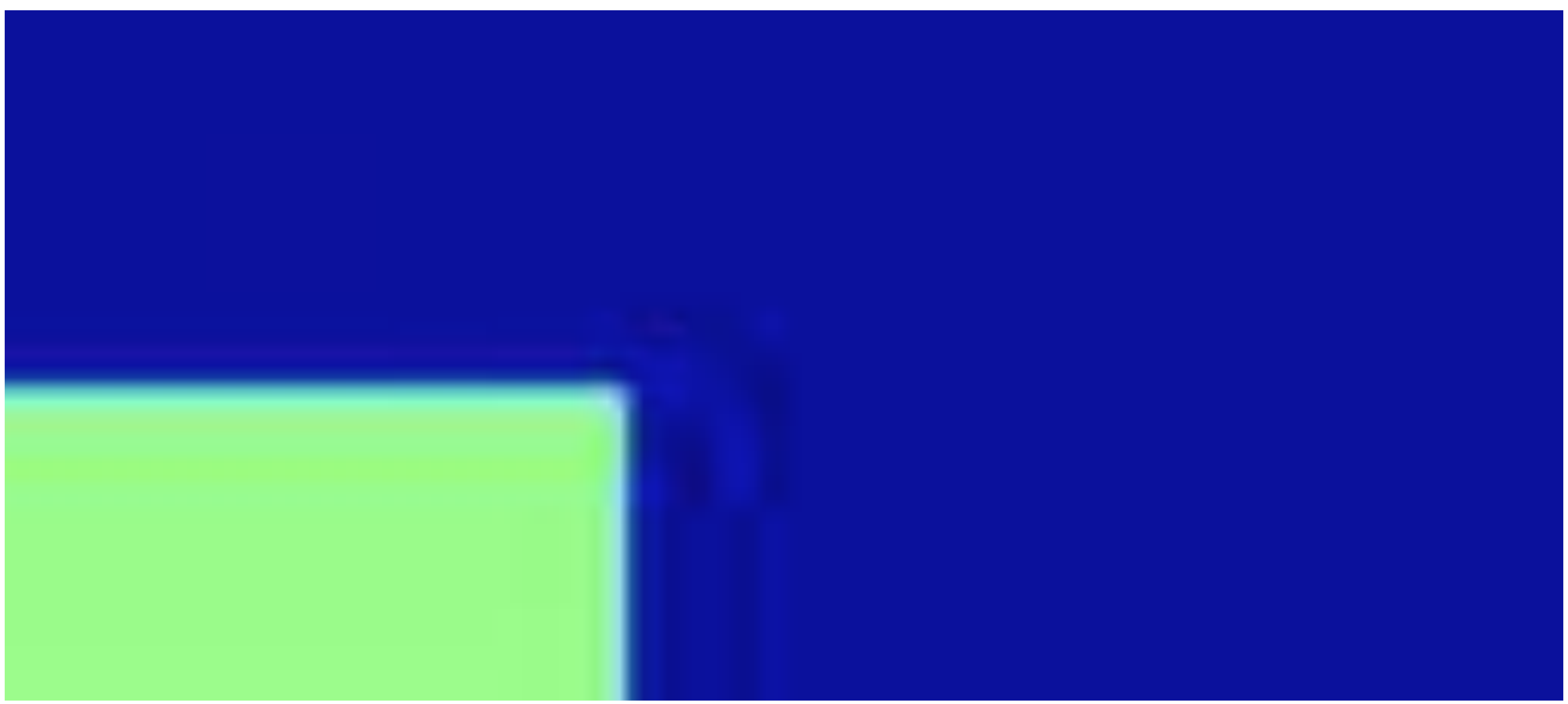
- Bingham rheology
- Granular μ(I) rheology
- Implementation in *Basilisk*

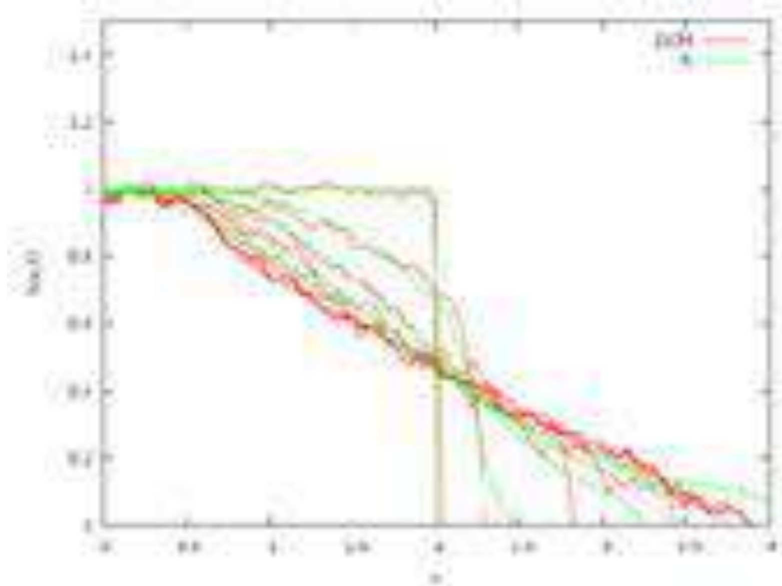
Granulars
- Example of column collapse
- Examples of silo

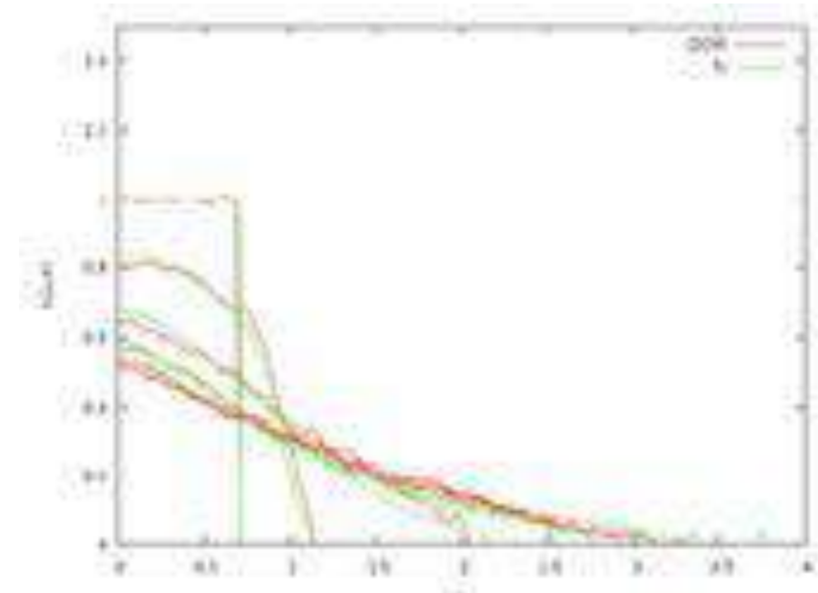
Bingham
- Hierarchy of models for Bingham

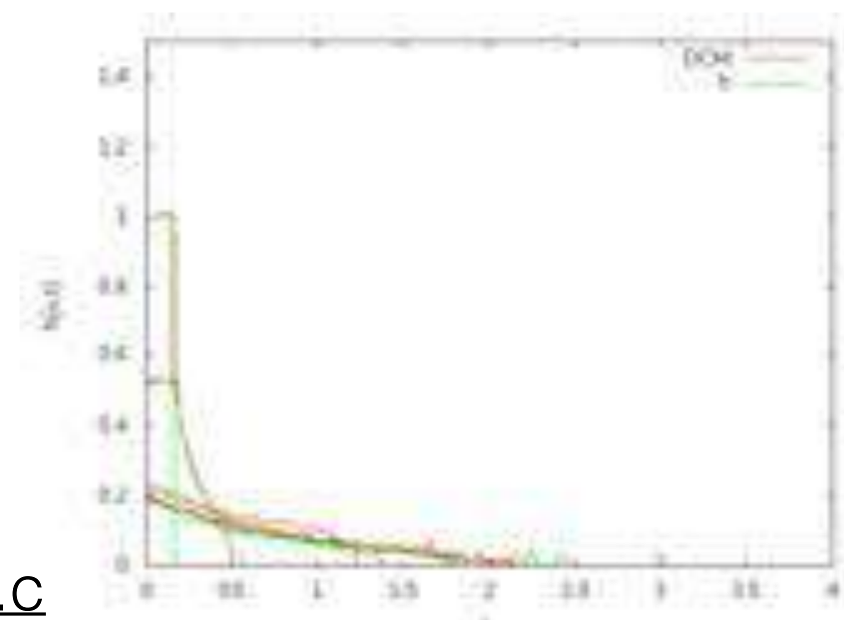
Collapse of columns simulation *Basilisk* $\mu(I)$

reproduce Lagrée Staron Popinet 2011



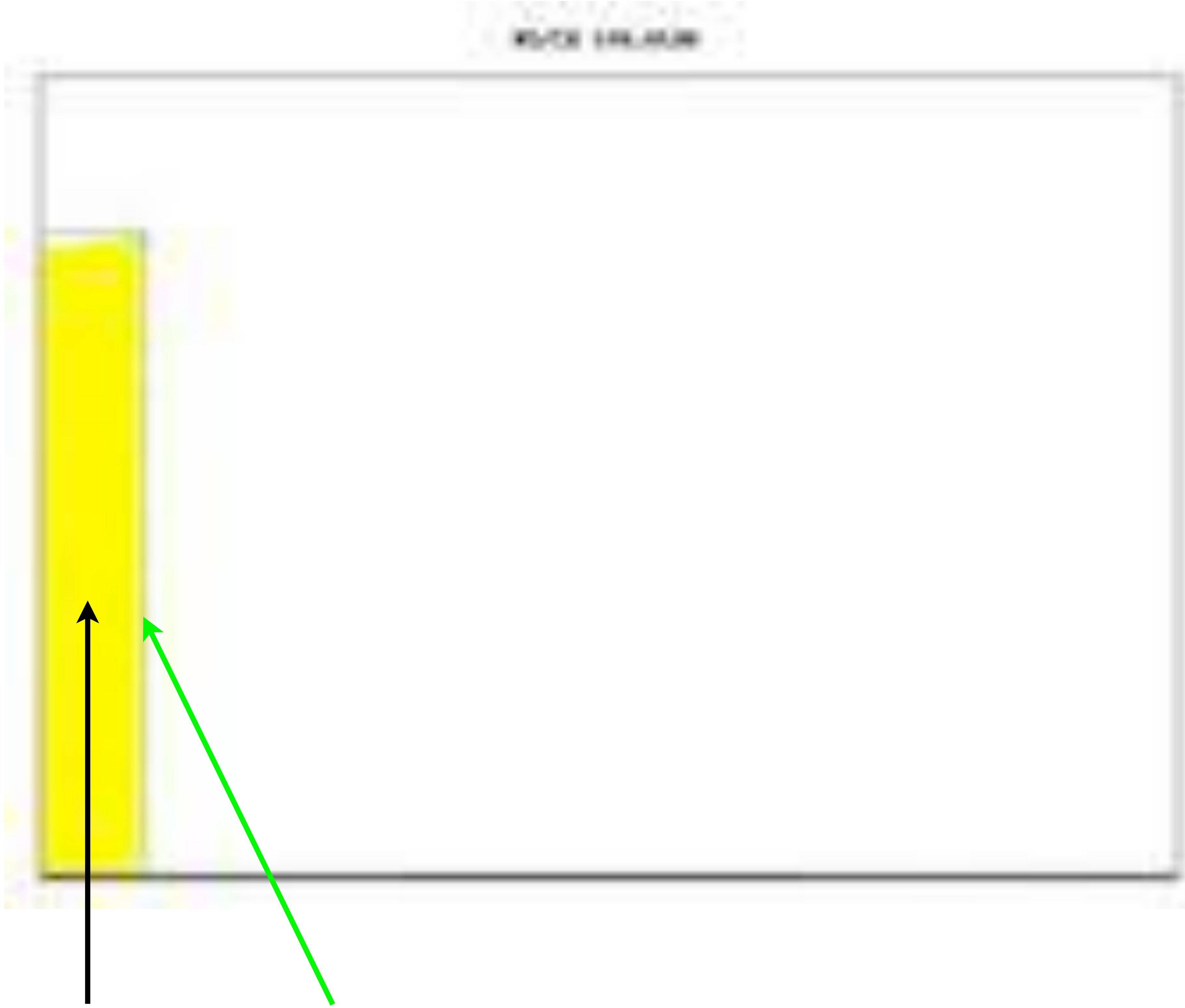






http://basilisk.fr/sandbox/M1EMN/Exemples/granular_sandglass.c

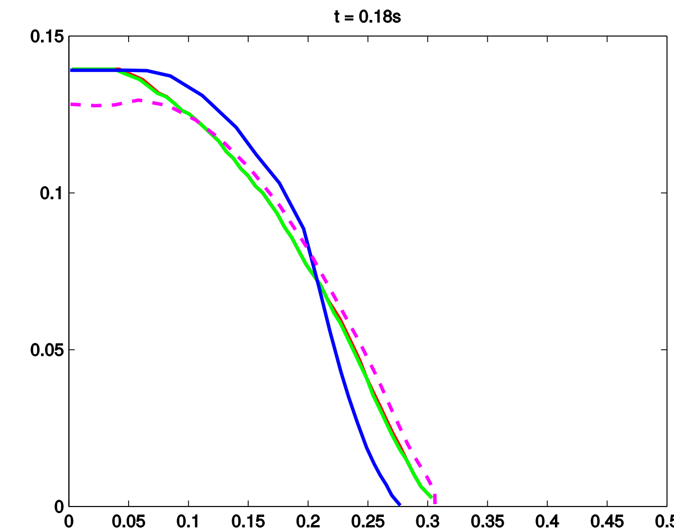
Collapse of columns simulation *Gerris* $\mu(I)$



DCM vs *Gerris* $\mu(I)$

Collapse of columns simulation *Gerris* $\mu(I)$

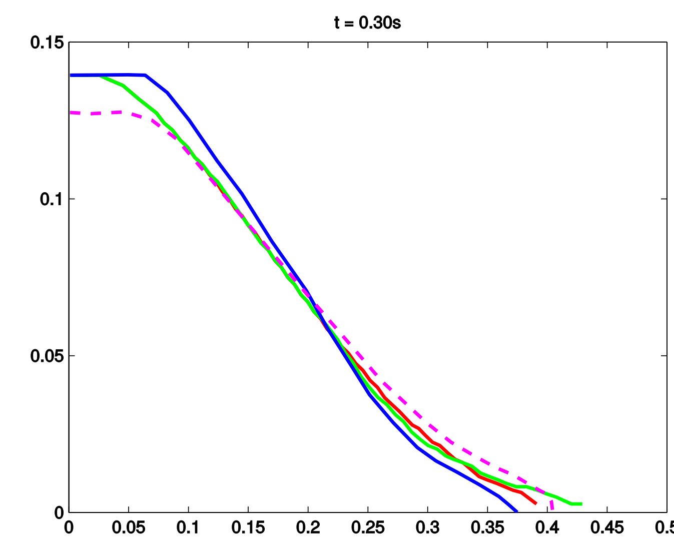
under work
with Sylvain Viroulet, Anne Mangeney IPGP



Solids are with friction at the wall

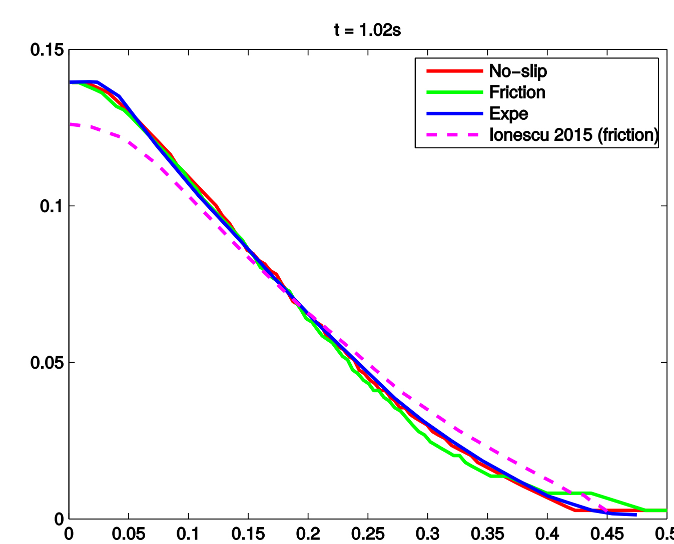
$$\tau = \mu_s p$$

implement solid friction at the wall
instead of no slip



Neumann condition, instead of no slip

$$\frac{\partial u}{\partial y}|_0 = \frac{\mu_s p}{\eta}$$



http://basilisk.fr/sandbox/M1EMN/Exemples/granular_column_muw.c

- Flow in a Hourglass Discharge from Hoppers

  simulation DCM (Lydie Staron)

- A well know experimental result:
  Hagen Beverloo constant discharge law

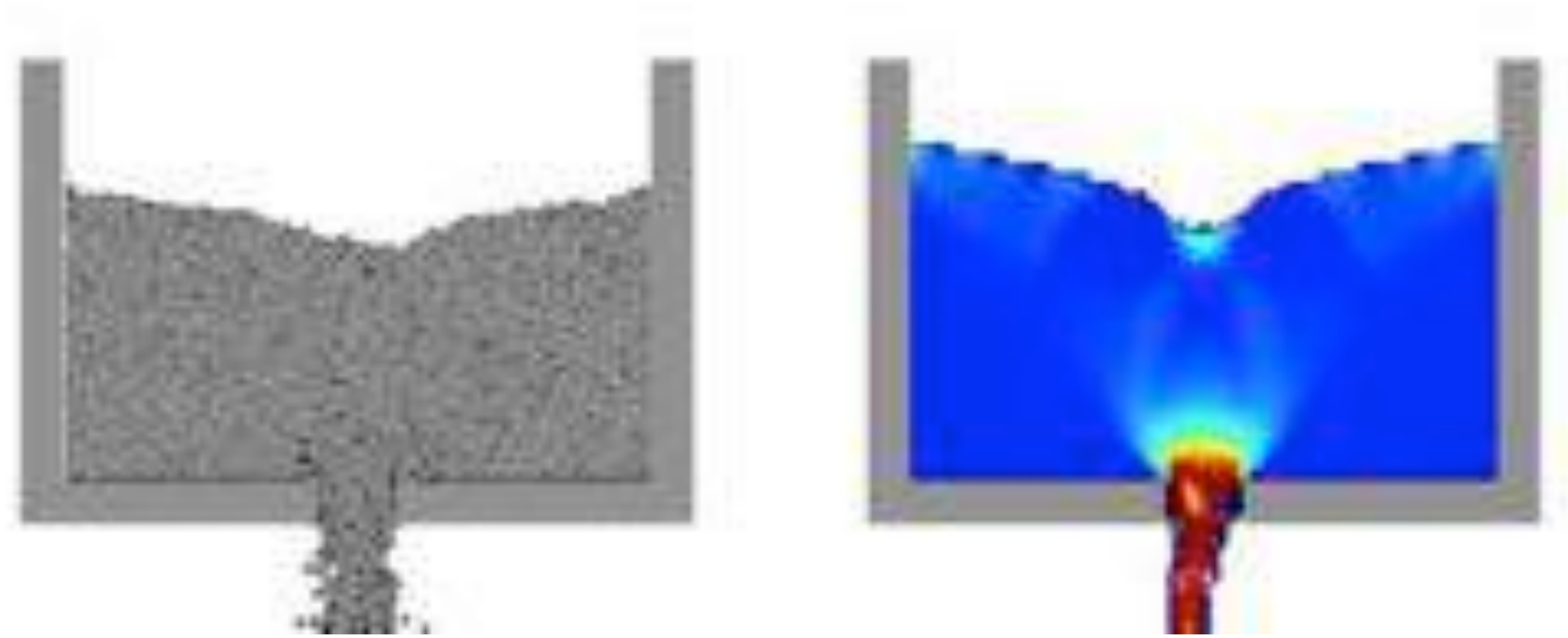
- Problem:

  Simulate the hour glass with discrete and continuum theories

- try to recover the Beverloo 1961 Hagen 1852 law from <u>discrete</u> and <u>continuum</u> simulations

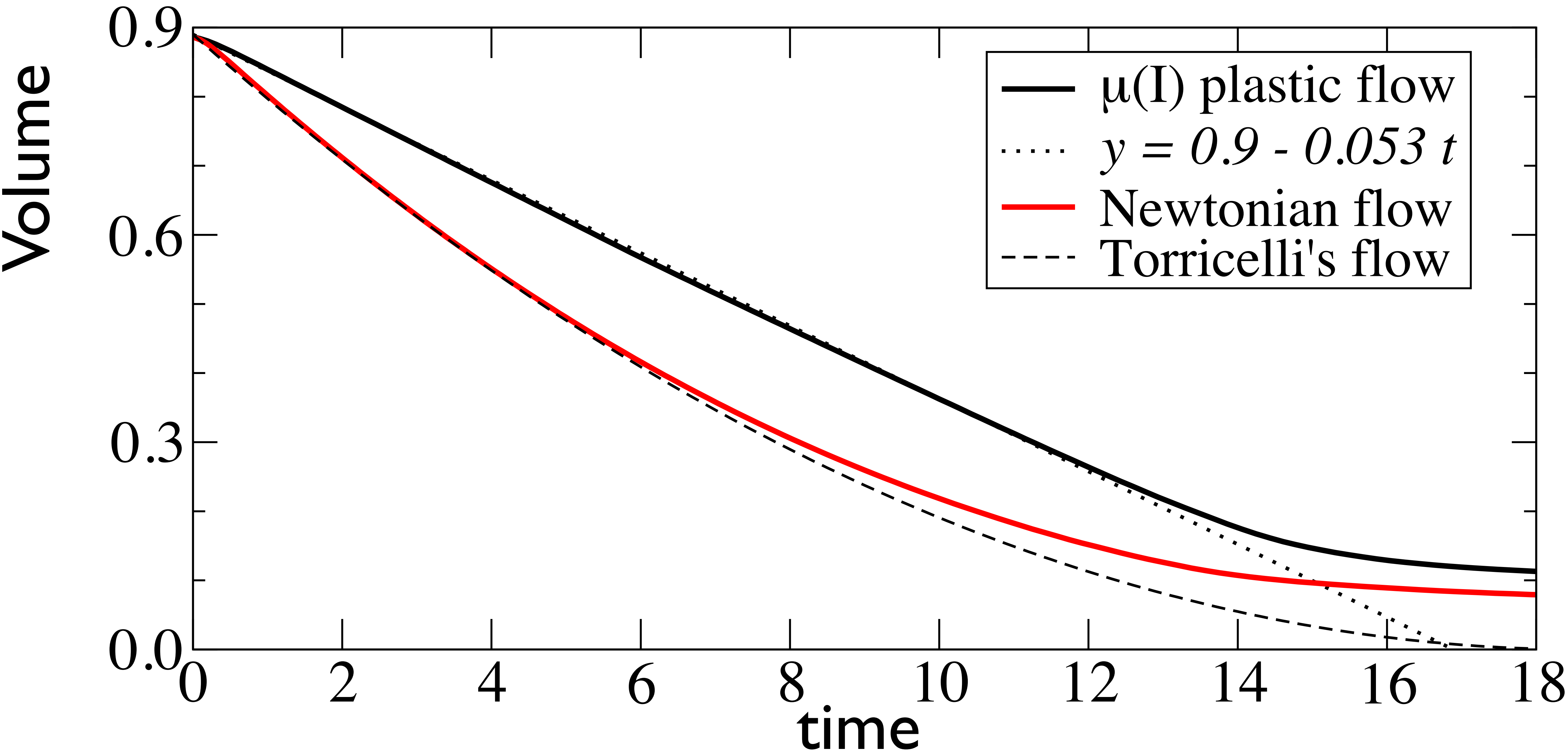
- Flow in a Hourglass Discharge from Hoppers

simulation discrete *vs* continuum



Evangelista Torricelli 1608 1647

- comparing Torricelli

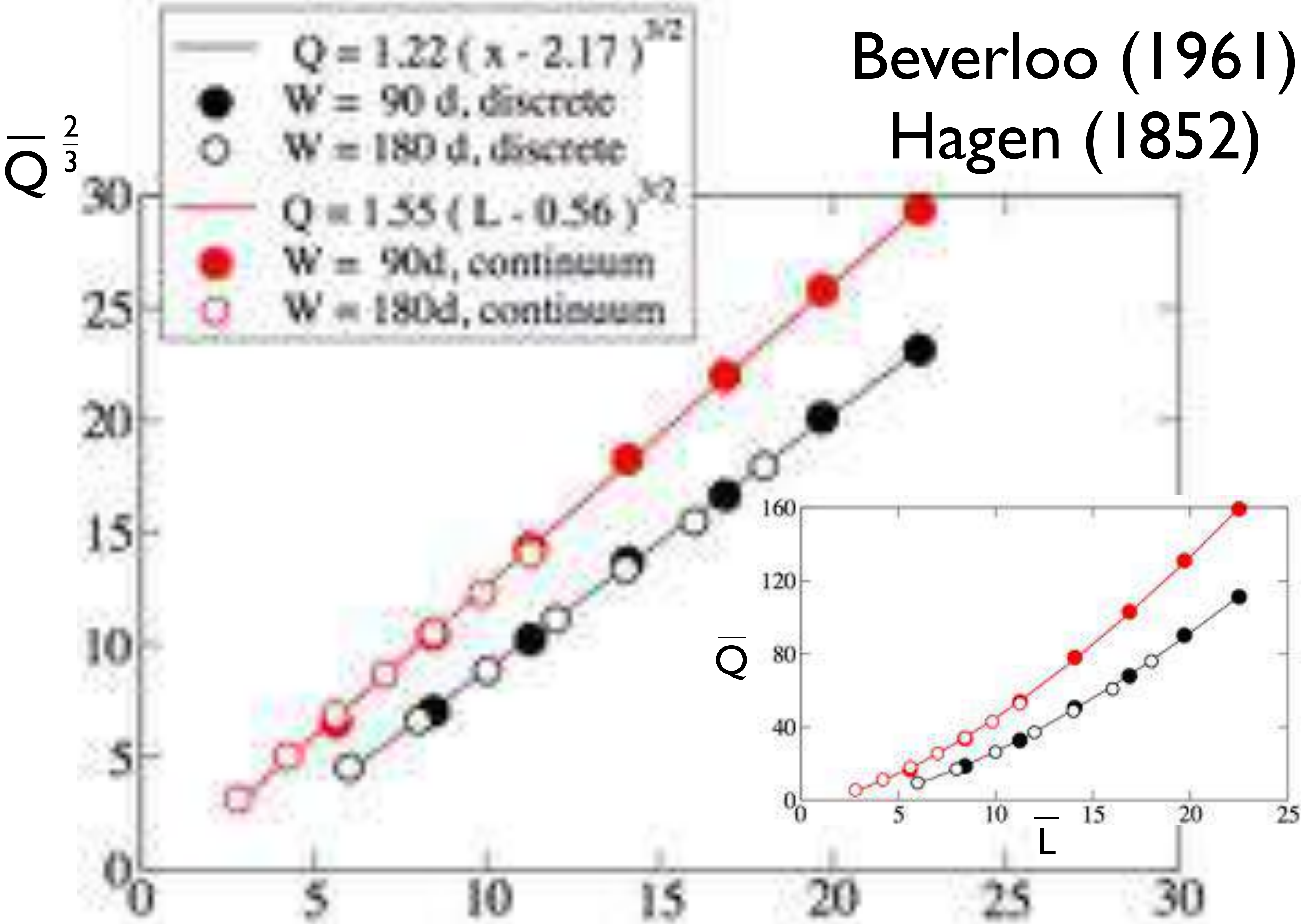


viscosity of the Newtonian flow extrapolated from the μ(I) near the orifice

- Flow in a Hourglass Discharge from Hoppers

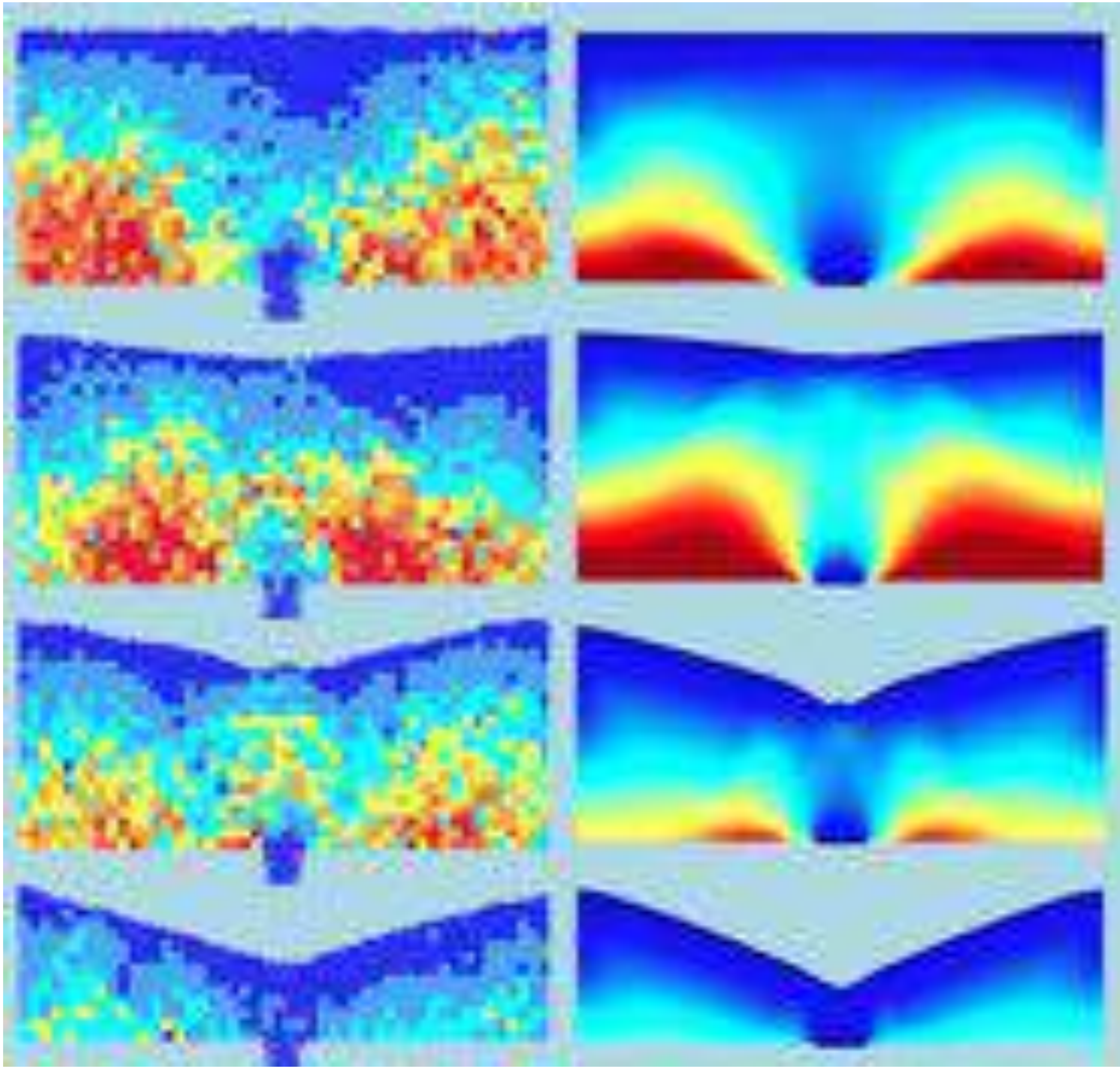
discrete *vs* continuum (a shift)

Beverloo (1961)
Hagen (1852)



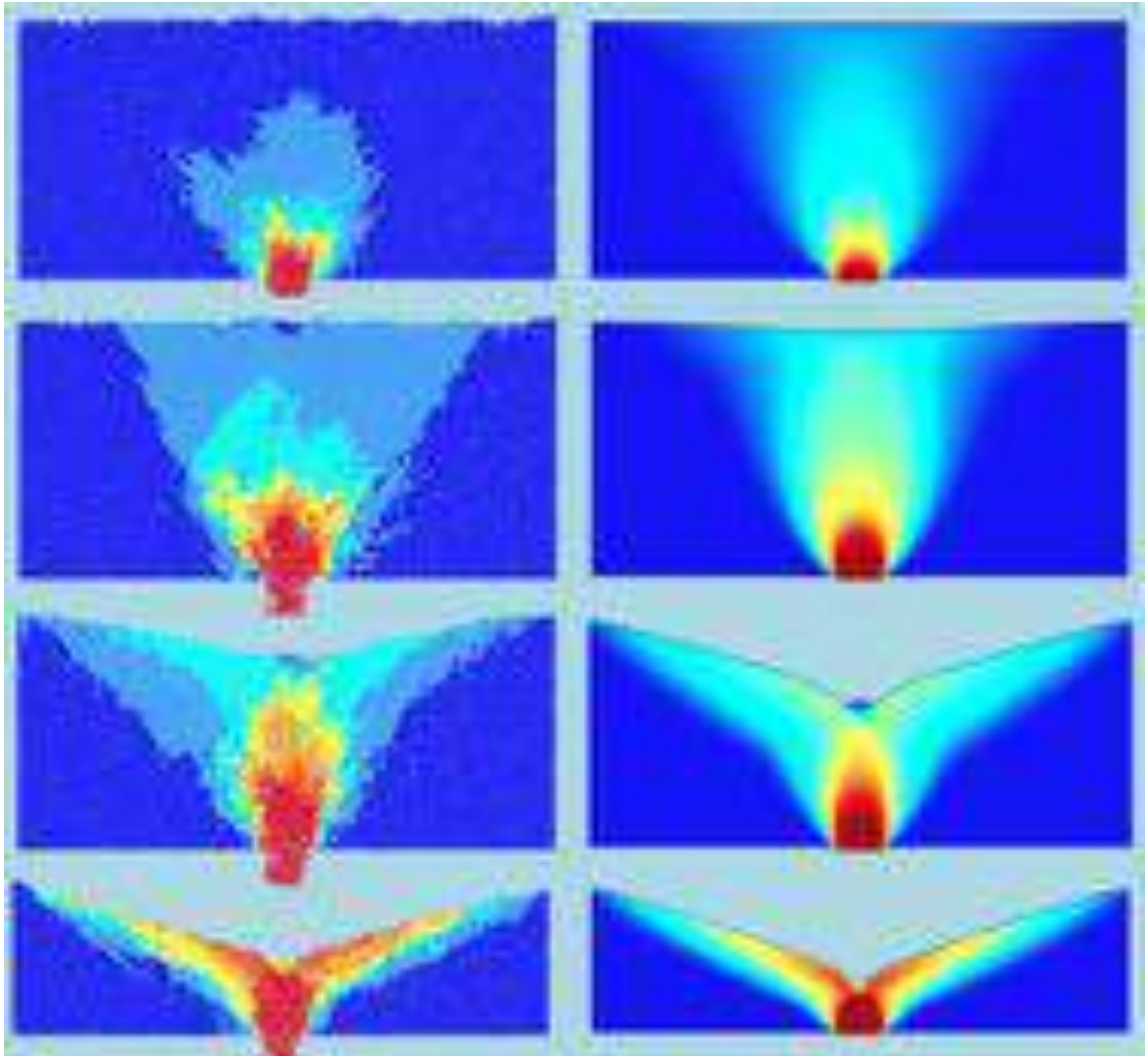
Width of the aperture

Staron Lagrée Popinet 2014



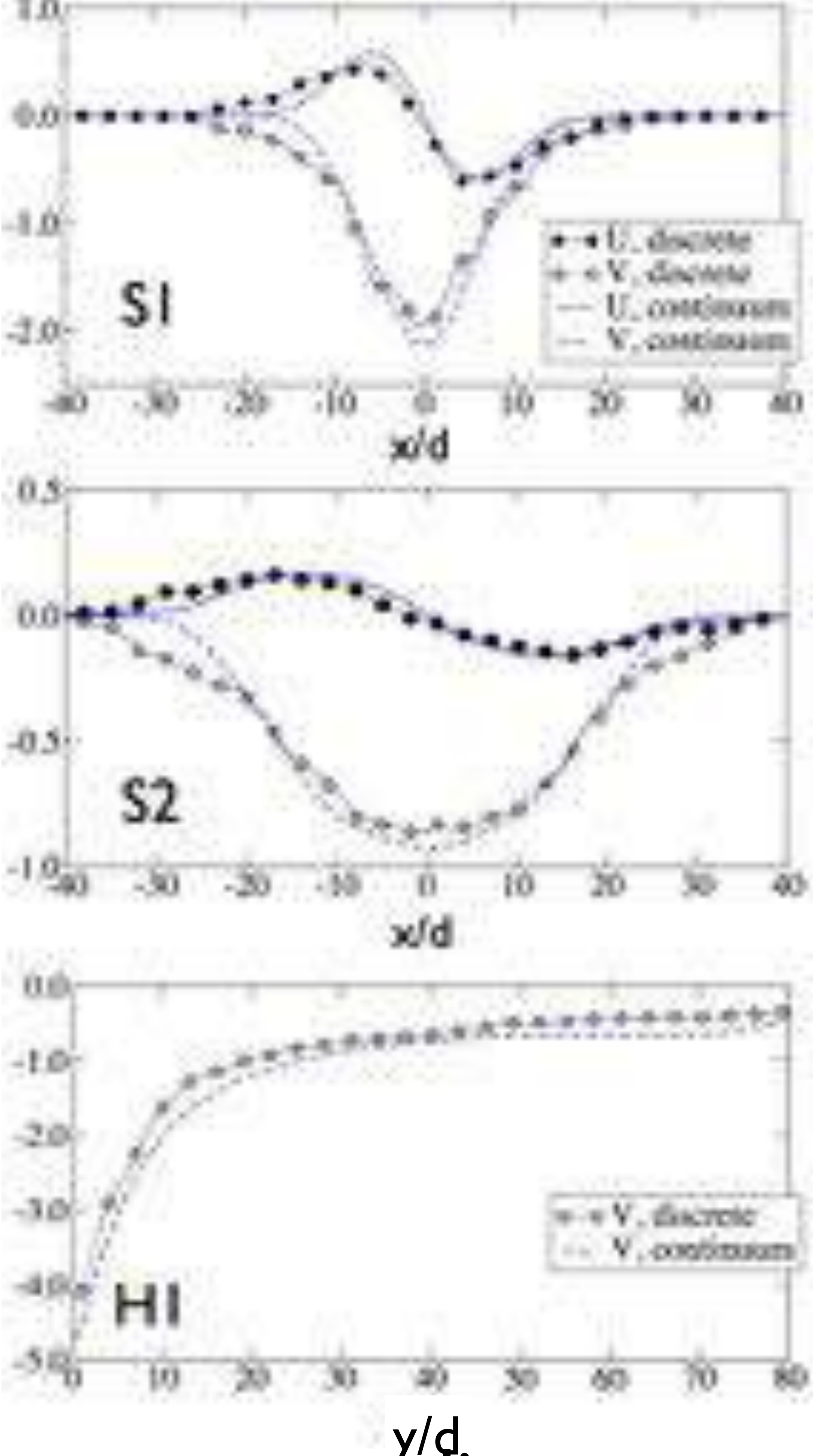
Staron Lagrée Popinet 2014

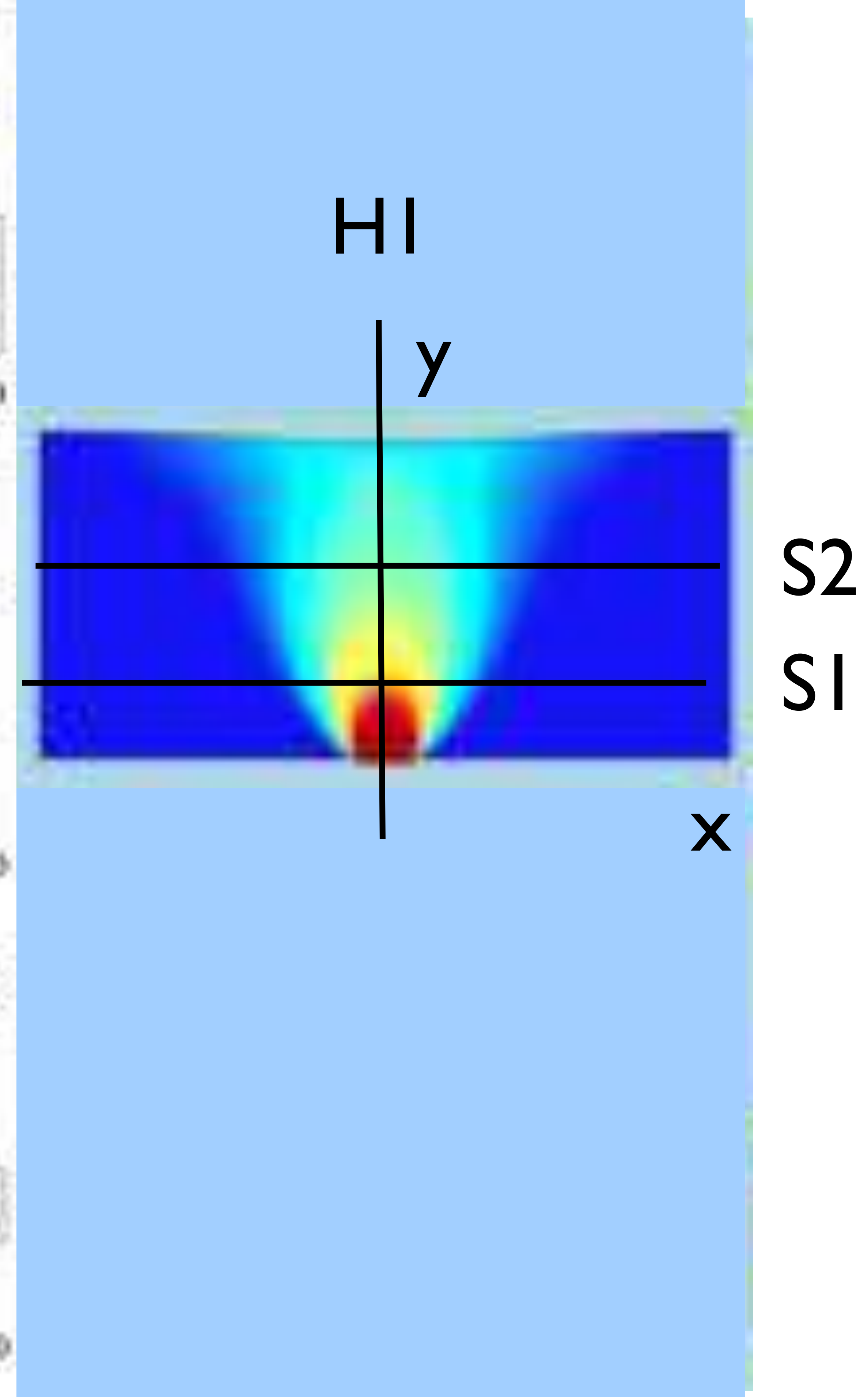
discrete *vs* continuum (at same rate)



Staron Lagrée Popinet 2014

discrete *vs* continuum (at same rate)





Staron Lagrée Popinet 2014

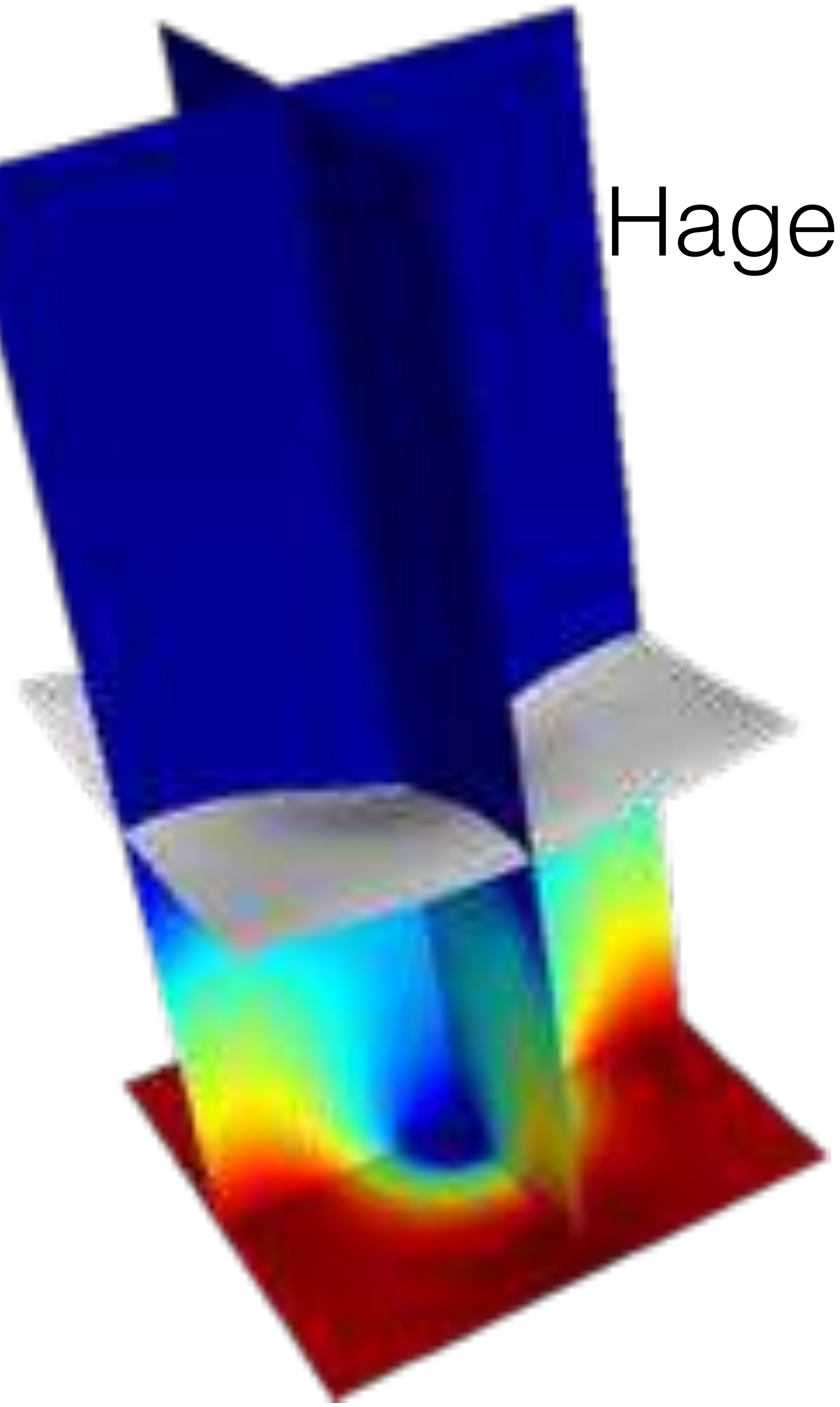
discrete *vs* continuum (at same rate)

Hagen 1852 Beverloo 1961

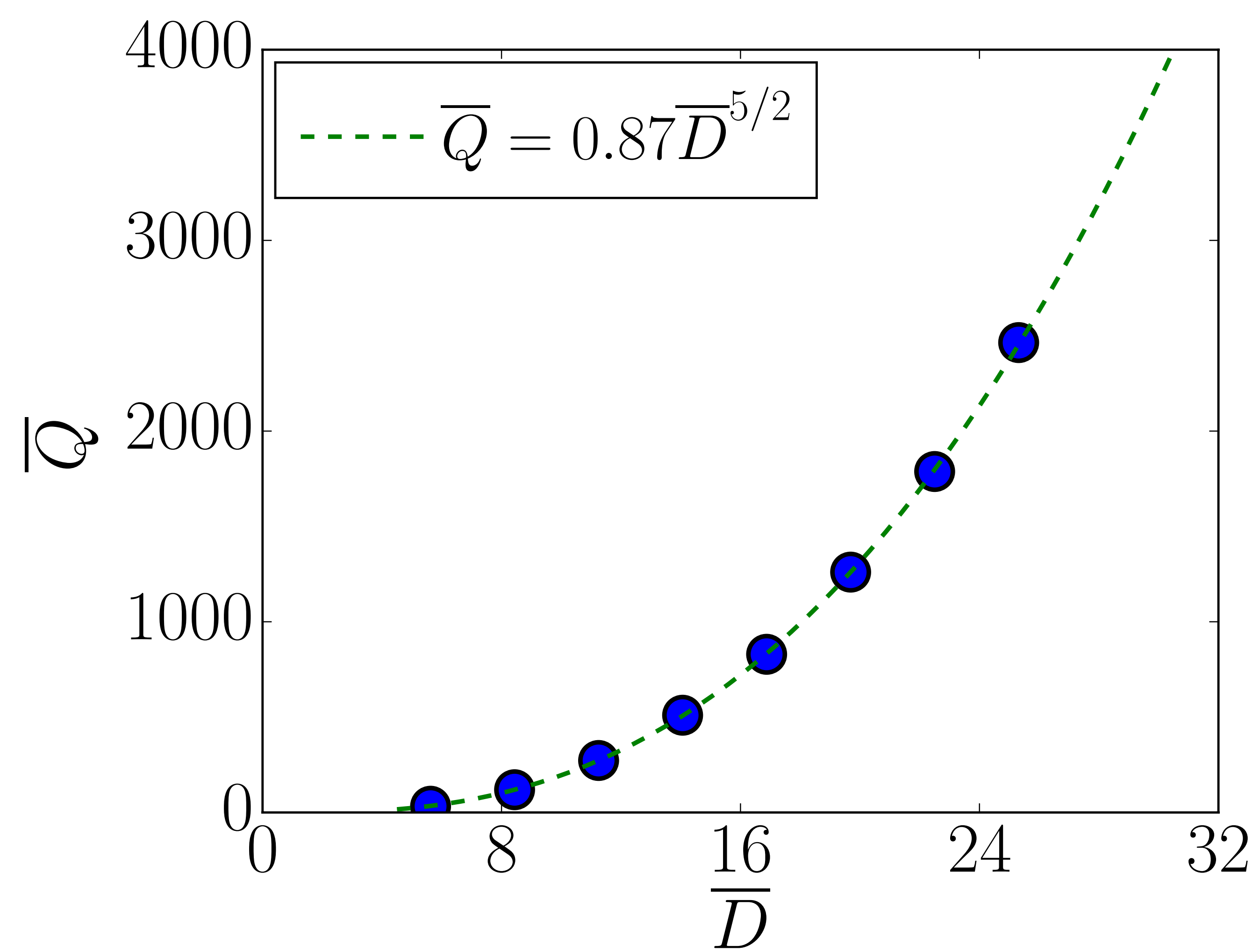
Gotthilf Hagen 1797-1884

Π-theorem

checked with NS-μ(I) in 2D and even in 3D (square geo.)



Hagen 1852 Beverloo 1961 $Q_{3D} \propto \rho\sqrt{gD^5}$



3D as Hele-Shaw approximation

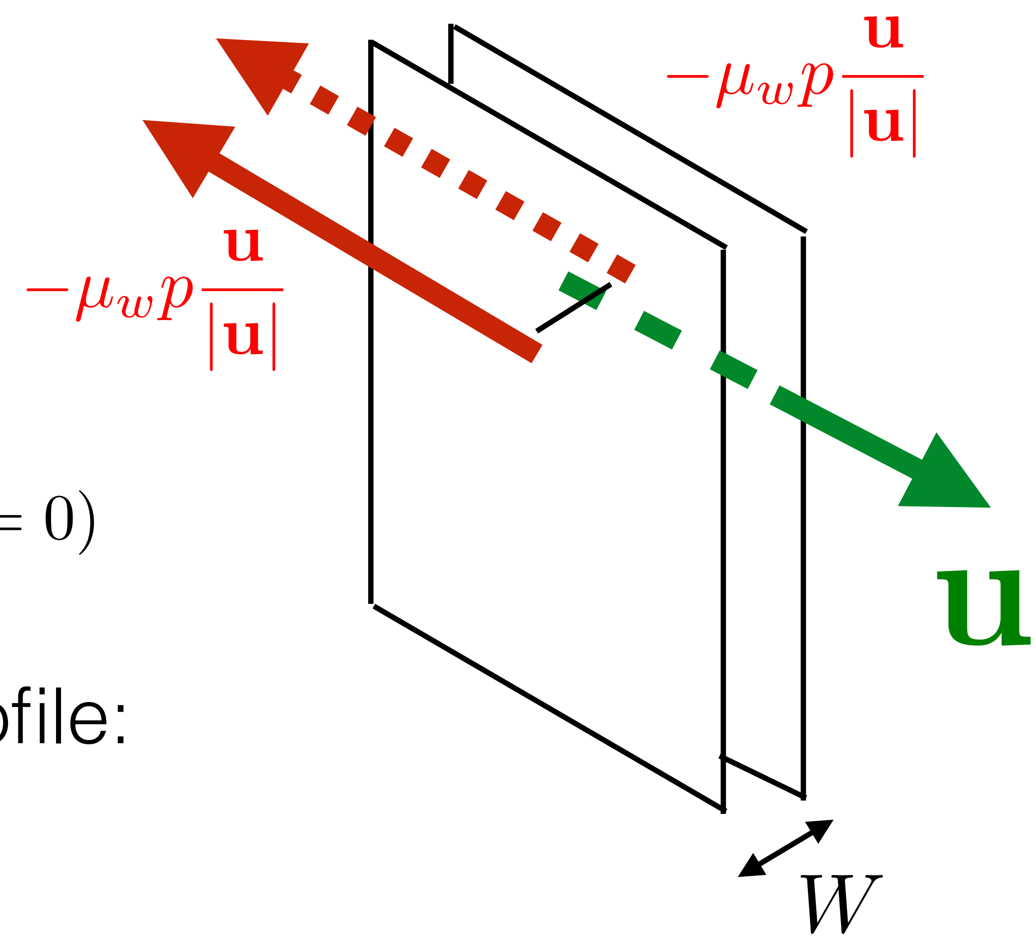
with Zhou Ruyer Aussillous

The 3D equations are averaged across the cell of thickness W

$$\mathbf{u} = (u, v, w)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot (2\eta \mathbf{D}) + \rho \mathbf{g}$$

$$\int \cdot \, dz$$



redefinition of velocity

$$\mathbf{u} = \left(\frac{1}{W} \int_{-W/2}^{W/2} u dz, \frac{1}{W} \int_{-W/2}^{W/2} v dz, w = 0\right)$$

suppose an almost transverse flat profile:
non linear closure coefficient is one
extra wall friction source term

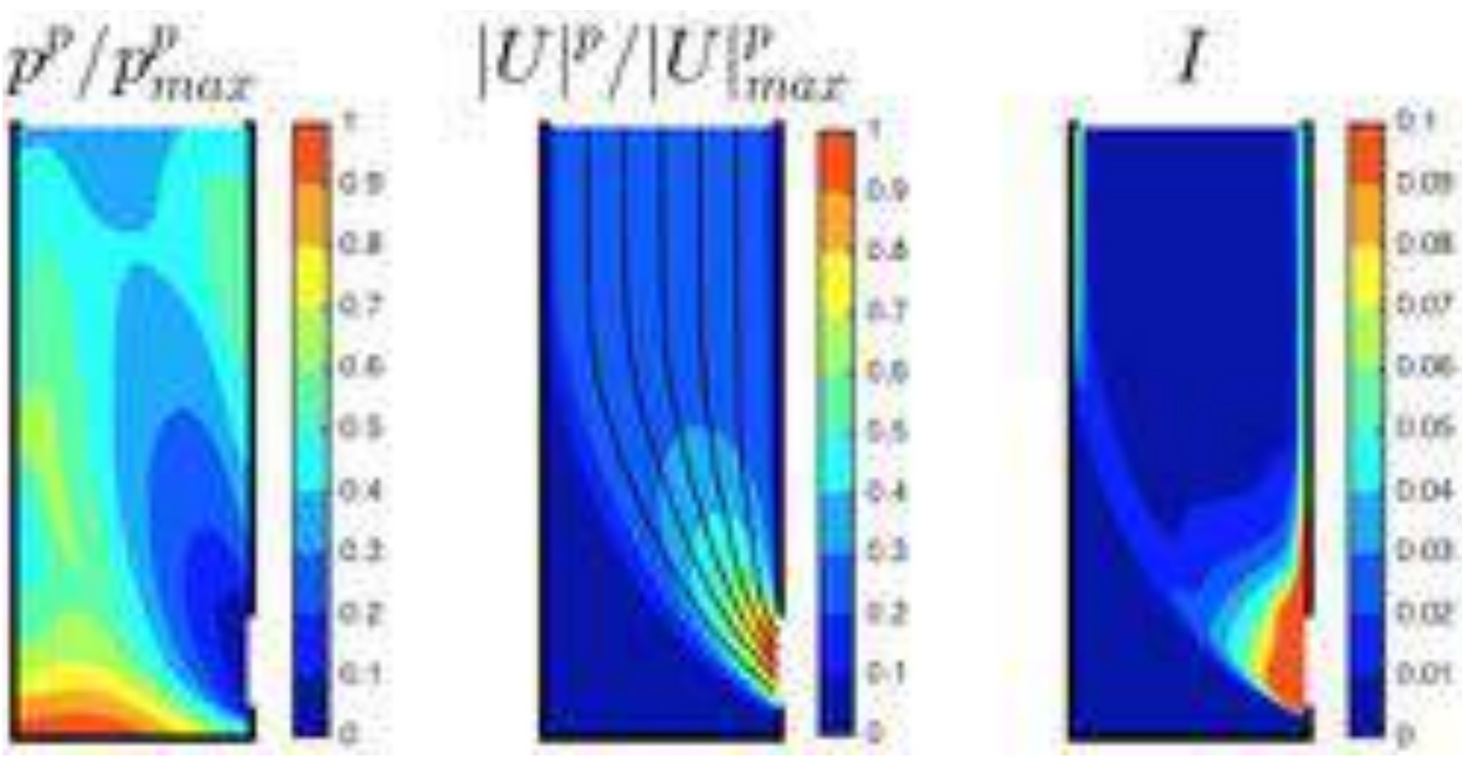
$$\mathbf{u} = (u, v)$$

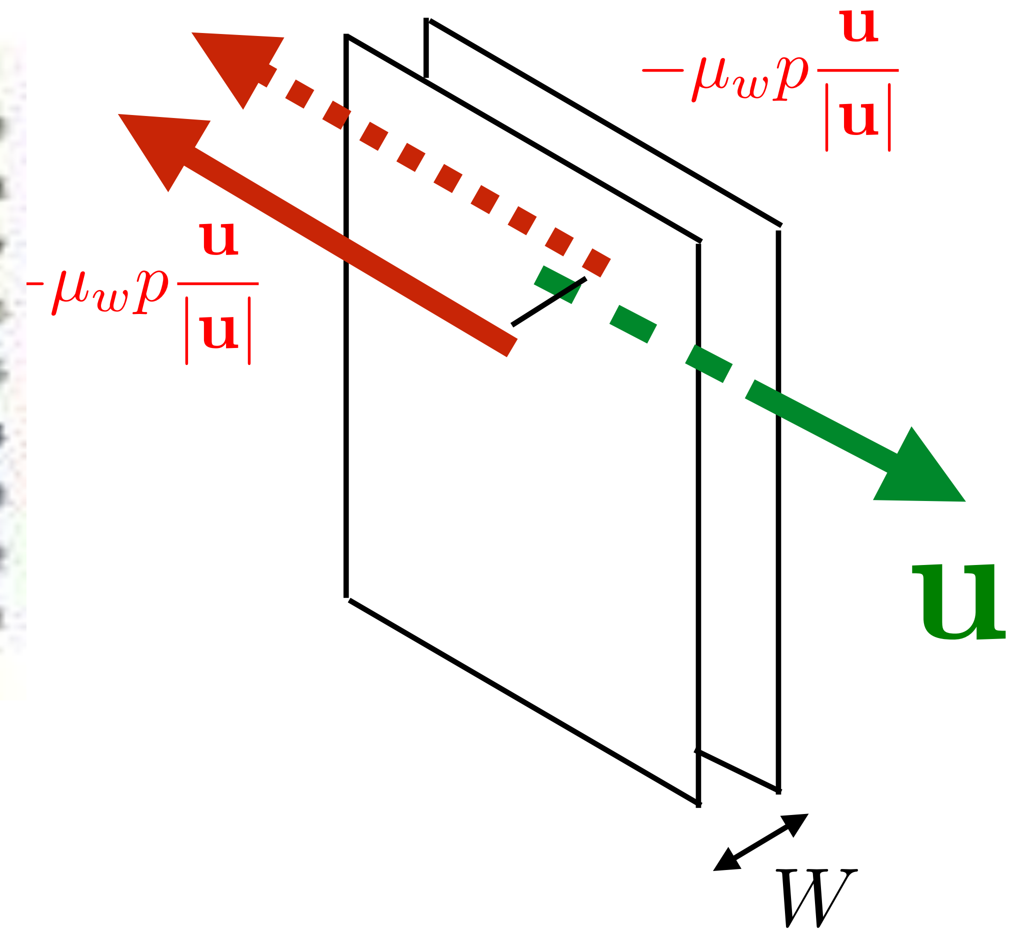
$$\nabla \cdot \mathbf{u} = 0, \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot (2\eta \mathbf{D}) + \rho \mathbf{g} - 2\mu_w \frac{p}{W} \frac{\mathbf{u}}{|\mathbf{u}|}$$

3D as Hele-Shaw approximation

with Zhou Ruyer Aussillous

The 3D equations are averaged across the cell of thickness W





$$\mathbf{u} = (u, v)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot (2\eta \mathbf{D}) + \rho \mathbf{g} - 2\mu_w \frac{p}{W} \frac{\mathbf{u}}{|\mathbf{u}|}$$

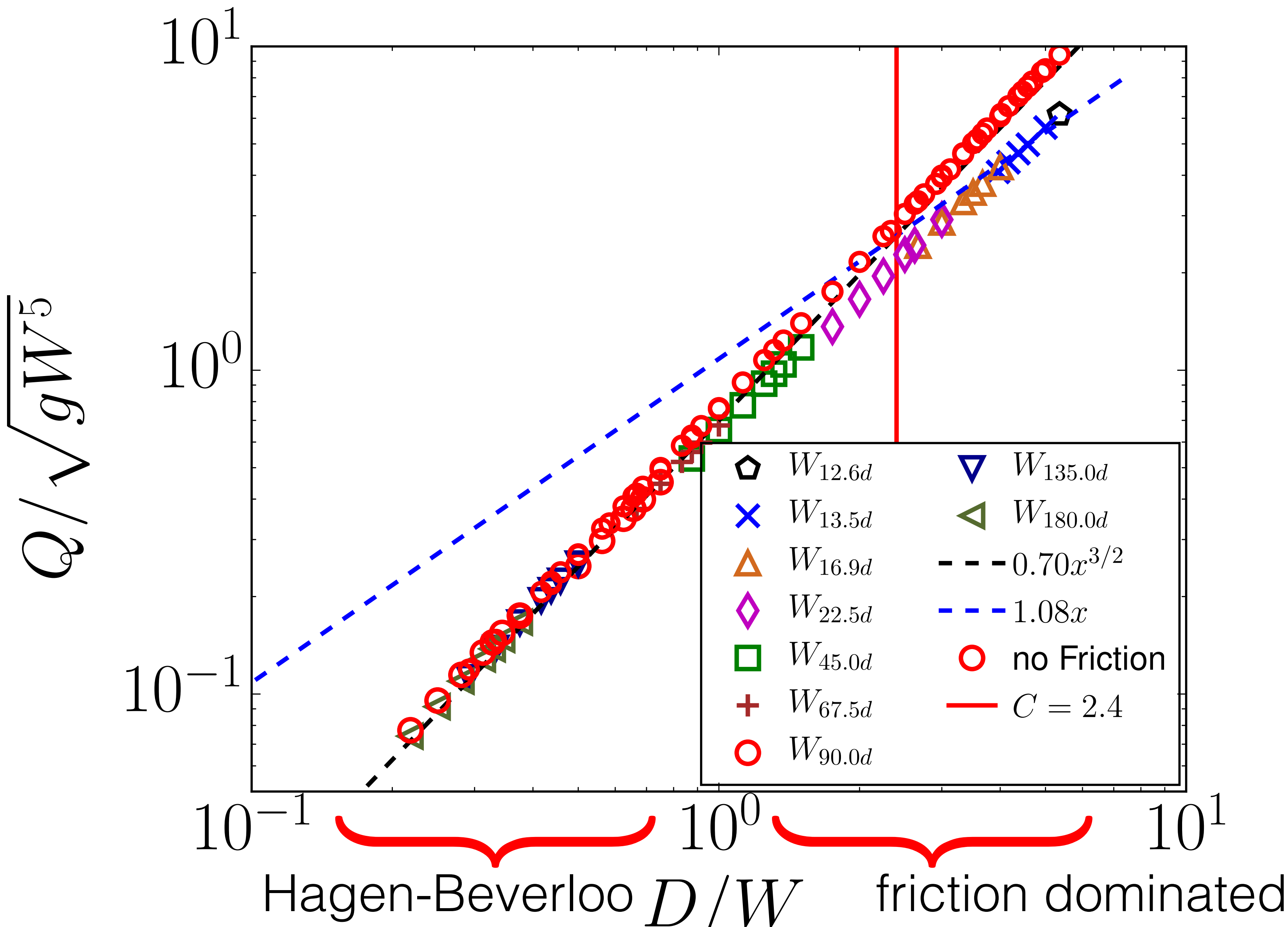
NS Hele-Shaw μ(I)

rescaling with: $$Q = \rho W^2 \sqrt{gW}\, \mathcal{F}(D/W)$$

large thickness $(D/W) \ll 1$

small thickness $(D/W) \gg 1$

plot as function of D/W



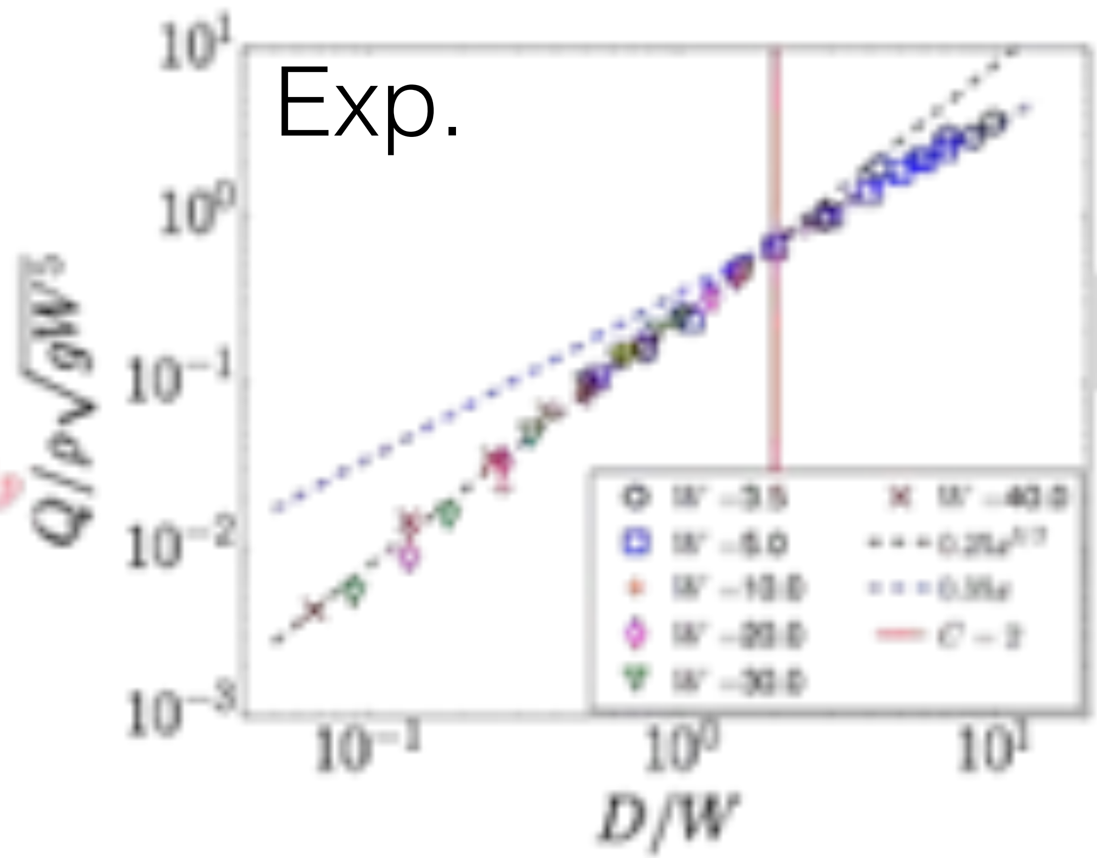
NS Hele-Shaw μ(I) vs Experiments

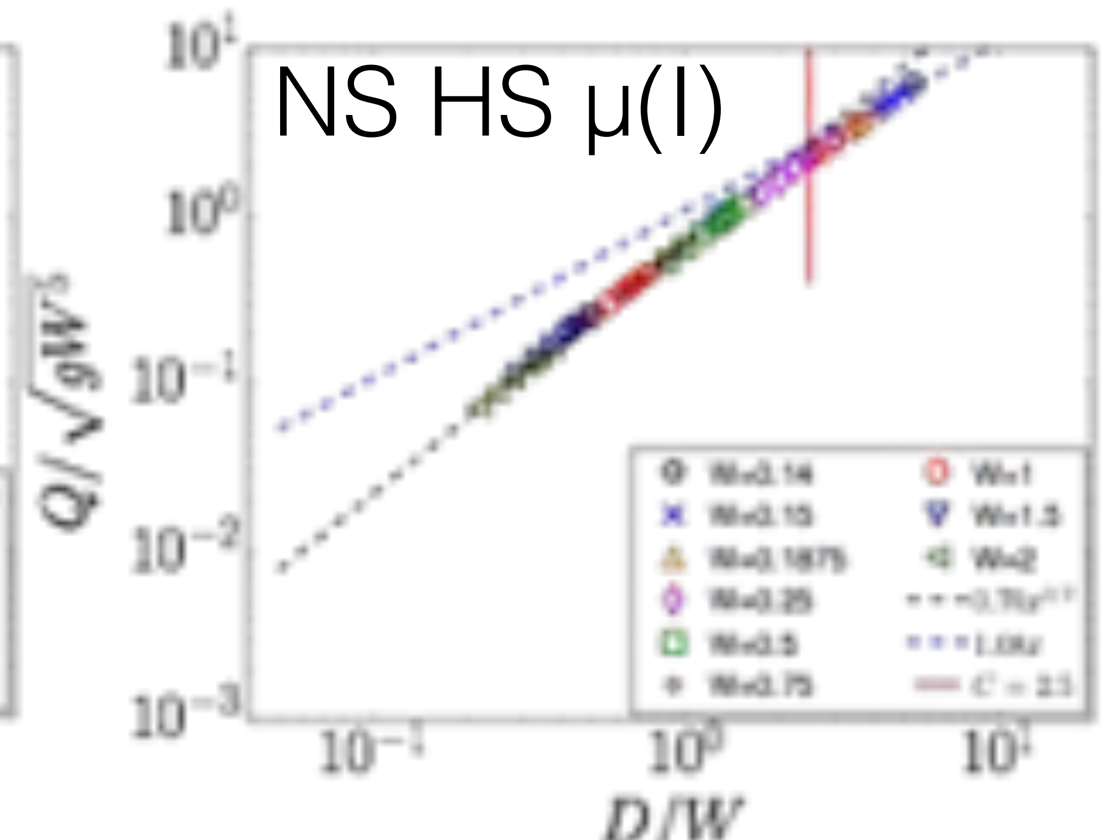
rescaling with: $Q = \rho W^2 \sqrt{gW}\, \mathcal{F}(D/W)$

large thickness $(D/W) \ll 1$

small thickness $(D/W) \gg 1$

plot as function of D/W



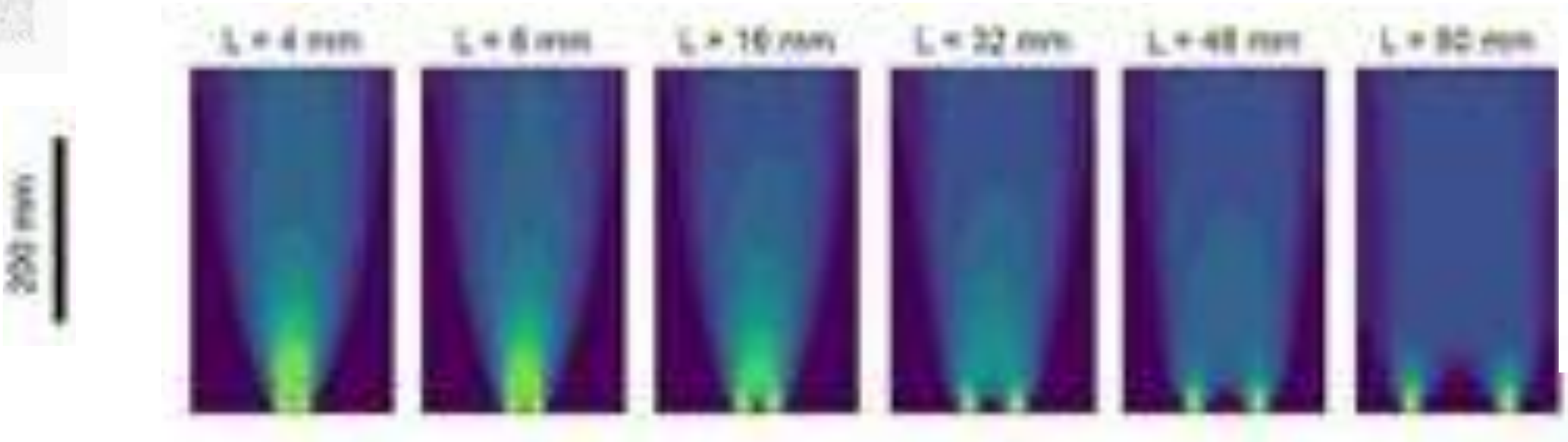


$(D/W) < 1, \quad Q \sim D^{3/2} W$ $\qquad$ $(D/W) > 1, \quad Q \sim W^{3/2} D$

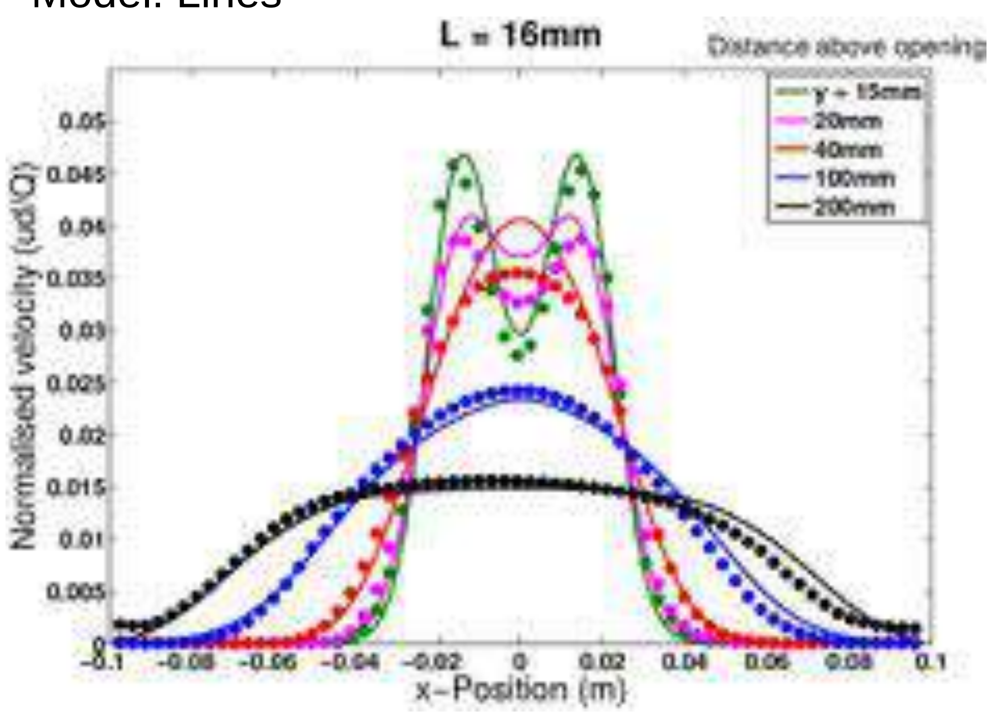
http://basilisk.fr/sandbox/M1EMN/Exemples/granular_sandglass_muw.c

two holes silo

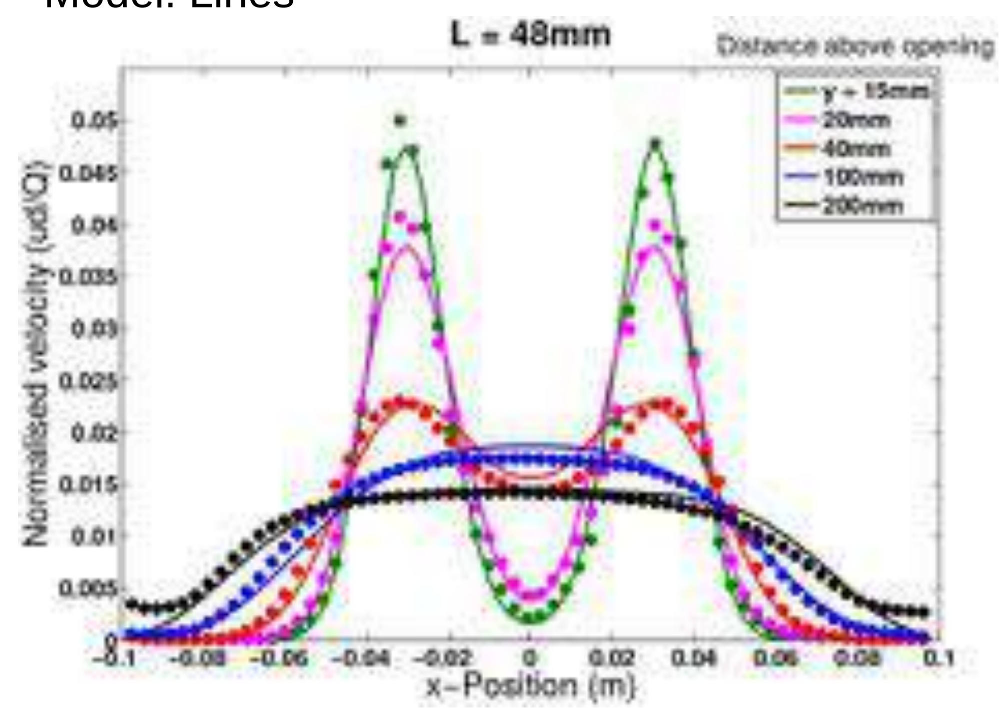
with Luke Fullard



Experiment: circle markers
Model: Lines



Experiment: circle markers
Model: Lines



- Bingham rheology
- Granular μ(I) rheology
- Implementation in *Basilisk*

Granulars

- Example of column collapse
- Examples of silo

Bingham

- Hierarchy of models for Bingham (and others)

Hierarchy of simplifications

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$\rho\Big(\frac{\partial v}{\partial y} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\Big) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \rho g \cos\theta$$

Full NS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

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Hierarchy of simplifications

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

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$$\cancel{\rho\Big(\frac{\partial v}{\partial y} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\Big)} = -\frac{\partial p}{\partial y} + \cancel{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}} - \rho g \cos\theta$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

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Hierarchy of simplifications

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$
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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$
$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

Reduced NS Prandtl
RNSP

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$
$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

Hierarchy of simplifications

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$\rho\Big(\frac{\partial v}{\partial y} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\Big) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \rho g \cos\theta$$

Full NS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

Reduced NS Prandtl
RNSP

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\cancel{\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big)} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

Hierarchy of simplifications

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

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Full NS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

Reduced NS Prandtl
RNSP

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\theta$$

lubrication

Hierarchy of simplifications

to be solved by *Basilisk*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$\rho\Big(\frac{\partial v}{\partial y} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\Big) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \rho g \cos\theta$$

Full NS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

Reduced NS Prandtl
RNSP

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[\frac{Uh}{3}(3 - \frac{Y}{h})] = 0$$

$$U = \frac{1}{\mu}(S - \frac{\partial h}{\partial x})\frac{Y^2}{2}$$

$$Y = max(h - \frac{B}{|S - \frac{\partial h}{\partial x}|}, 0)$$

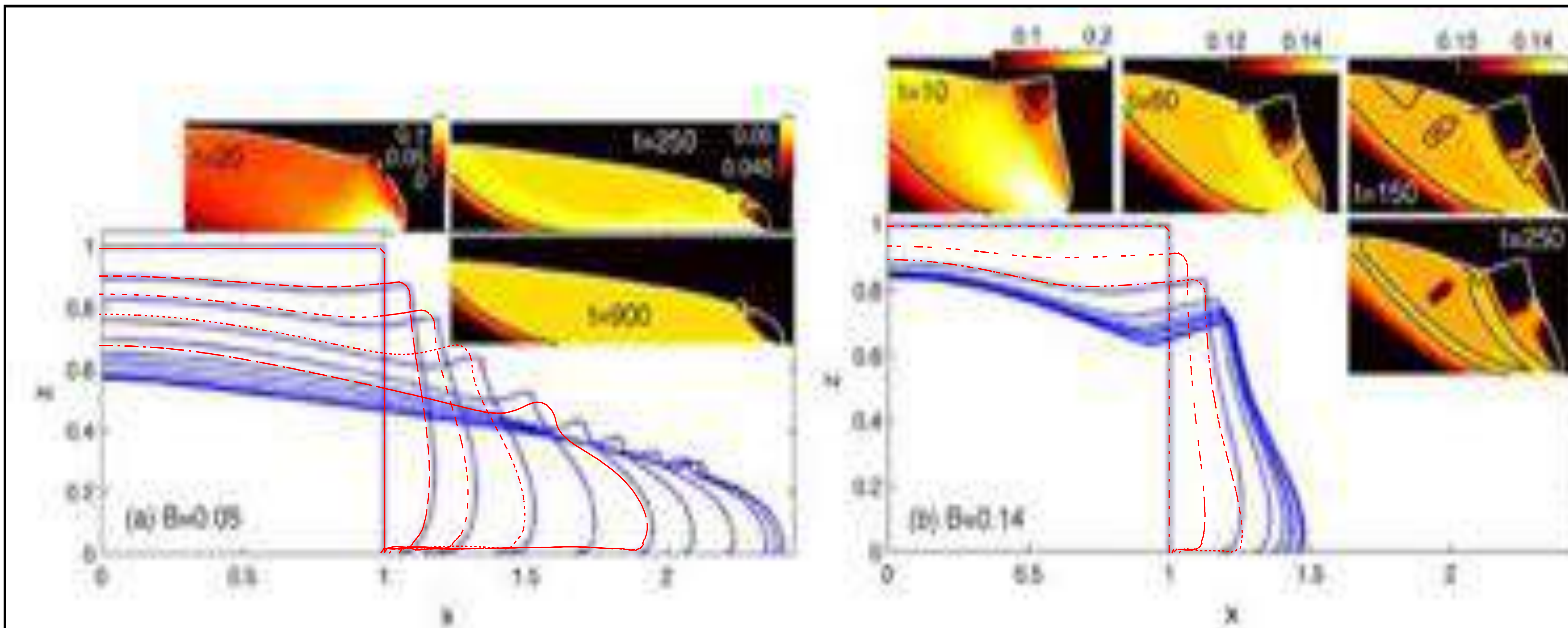
lubrication

Full NS Bingham

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$\rho\Big(\frac{\partial v}{\partial y} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y}\Big) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \rho g \cos\theta$$



http://basilisk.fr/sandbox/M1EMN/Exemples/column_SCC.c

Reduced NS Prandtl, RNSP Bingham

with Francesco De Vita

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\Big(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y}$$

AN EXCHANGING MASS MULTILAYER SAINT-VENANT SYSTEM

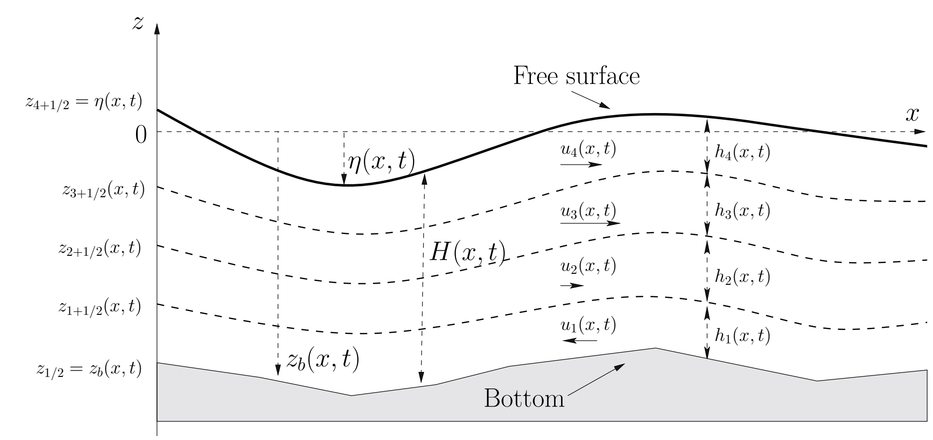


FIGURE 2. Notations for the multilayer approach.

"Multilayer" resolution:
integrate on fraction of height

coupling of several shallow water with interaction
(Audusse Bristeau Perthame Sainte-Marie 2011)

$$\begin{aligned} \frac{\partial h_\alpha}{\partial t} + \frac{\partial h_\alpha u_\alpha}{\partial x} &= G_{\alpha+1/2} - G_{\alpha-1/2}, \\ \frac{\partial h_\alpha u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha \left\langle u^2 \right\rangle_\alpha\right) + g h_\alpha \frac{\partial H}{\partial x} &= -g h_\alpha \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}. \end{aligned}$$

http://basilisk.fr/sandbox/M1EMN/Exemples/viscous_collapse_ML.c

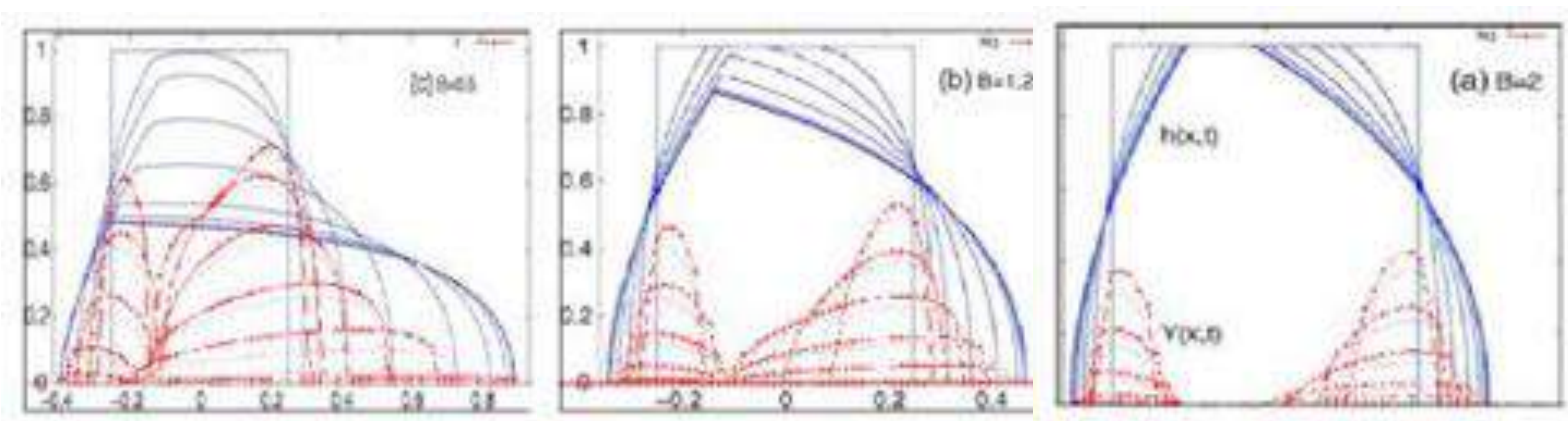
http://basilisk.fr/sandbox/M1EMN/Exemples/bingham_collapse_ML.c

lubrication   Bingham

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[\frac{Uh}{3}(3 - \frac{Y}{h})] = 0$$

$$U = \frac{1}{\mu}(S - \frac{\partial h}{\partial x})\frac{Y^2}{2}$$

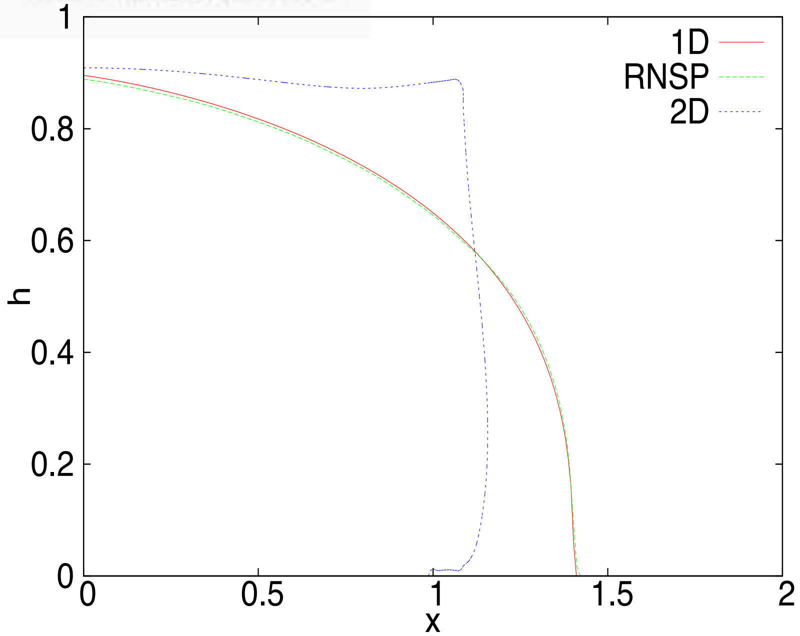
$$Y = max(h - \frac{B}{|S - \frac{\partial h}{\partial x}|}, 0)$$

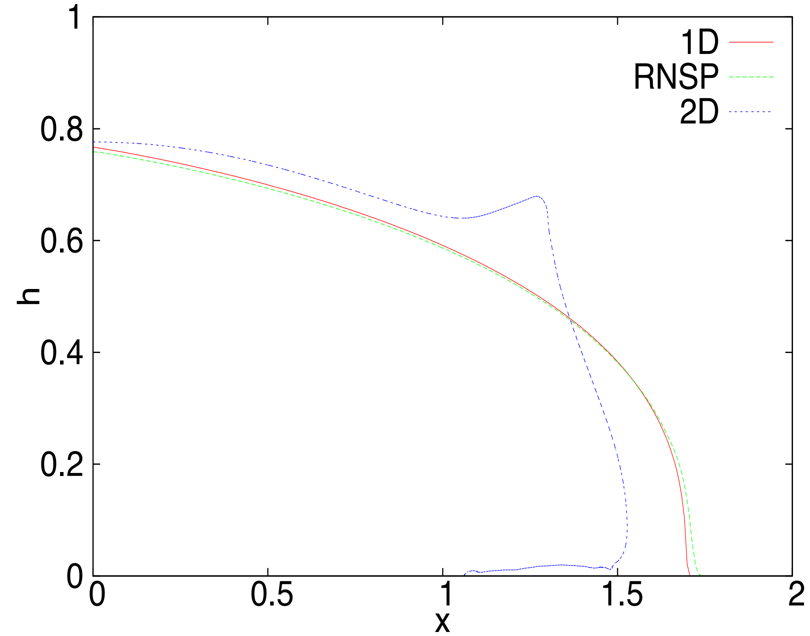


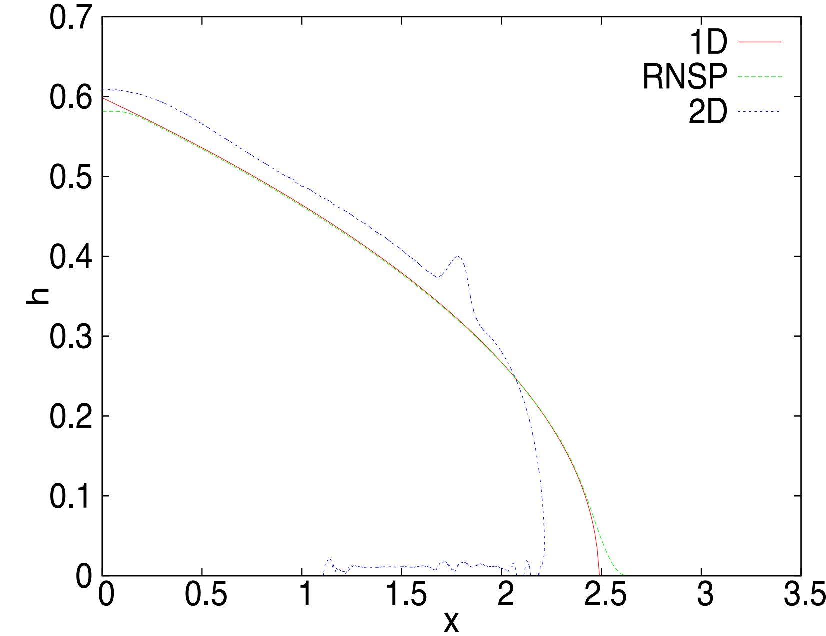
- N. Balmforth, R. Craster, A. Rust, R. Sassi "Viscoplastic flow over an inclined surface", J. Non-Newtonian Fluid Mech. 139 (2006) 103–127
- K.F. Liu, C.C. Mei, "Slow spreading of Bingham fluid on an inclined plane", J. Fluid Mech. 207 (1989) 505–529.

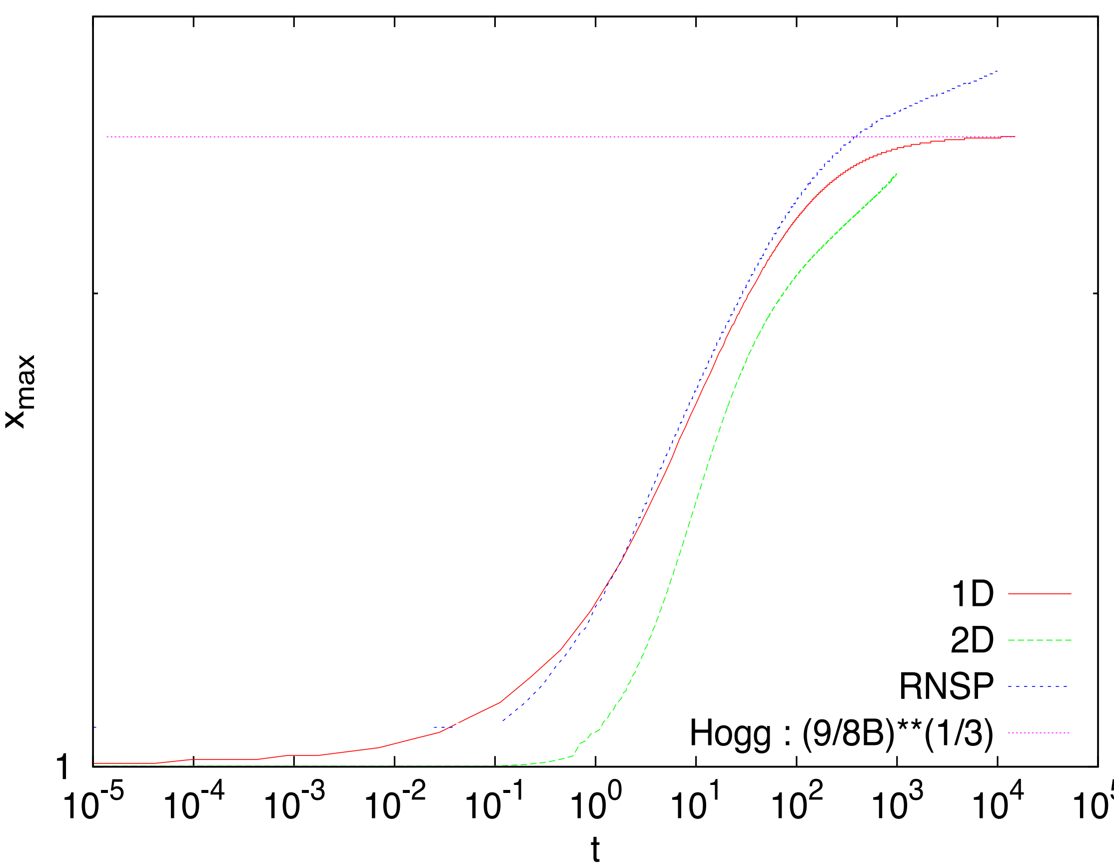
http://basilisk.fr/sandbox/M1EMN/Exemples/bingham_collapse_noSV.c

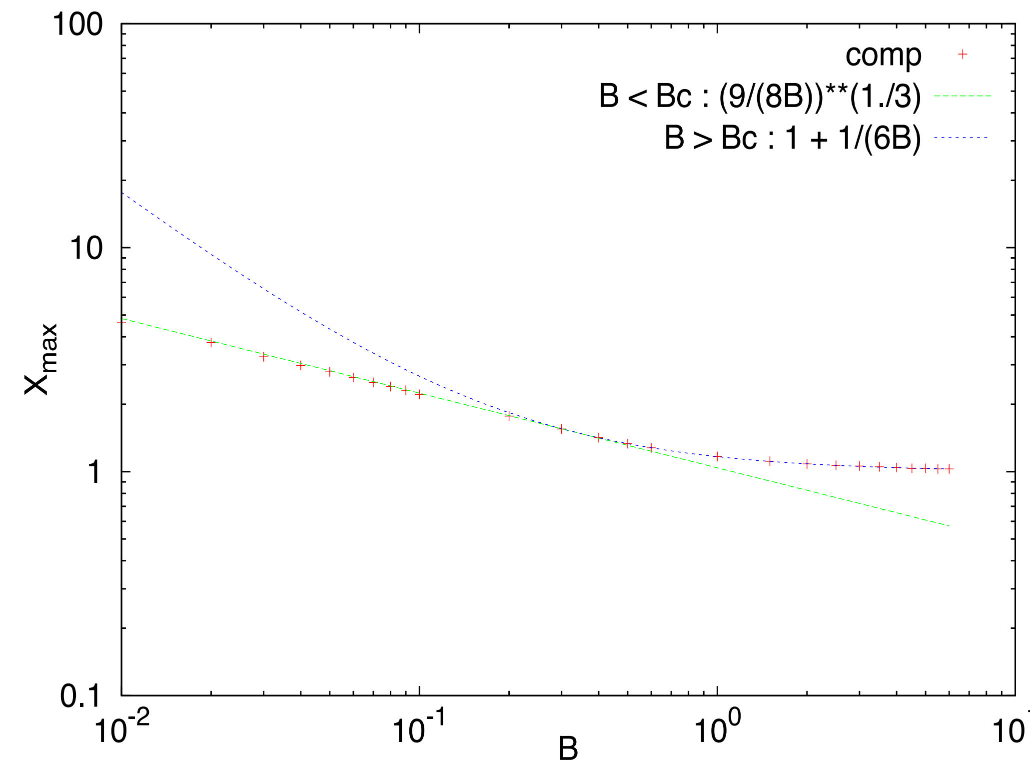
Bingham compare 2D RNSP 1D











Hogg & Matson JNFM 2009

Conclusion: a simple class of non newtonian flows solved with *Basilisk*

compared experiments, discrete and continuum simulations

http://basilisk.fr/sandbox/M1EMN/Exemples/column_SCC.c
http://basilisk.fr/sandbox/M1EMN/Exemples/granular_column_muw.c
http://basilisk.fr/sandbox/M1EMN/Exemples/granular_sandglass.c
http://basilisk.fr/sandbox/M1EMN/Exemples/granular_sandglass_muw.c

+ shallow water Savage Hutter on the web

http://basilisk.fr/sandbox/M1EMN/Exemples/front_poul_ed.c

comparisons
-2D
-RNSP Multilayer/
-1D (integral)

http://basilisk.fr/sandbox/M1EMN/Exemples/bingham_collapse_noSV.c
http://basilisk.fr/sandbox/M1EMN/Exemples/viscous_collapse_ML.c
http://basilisk.fr/sandbox/M1EMN/Exemples/bingham_collapse_ML.c

the model for the pertinent level of simplification

Jean Le Rond ∂'Alembert
1717 1783

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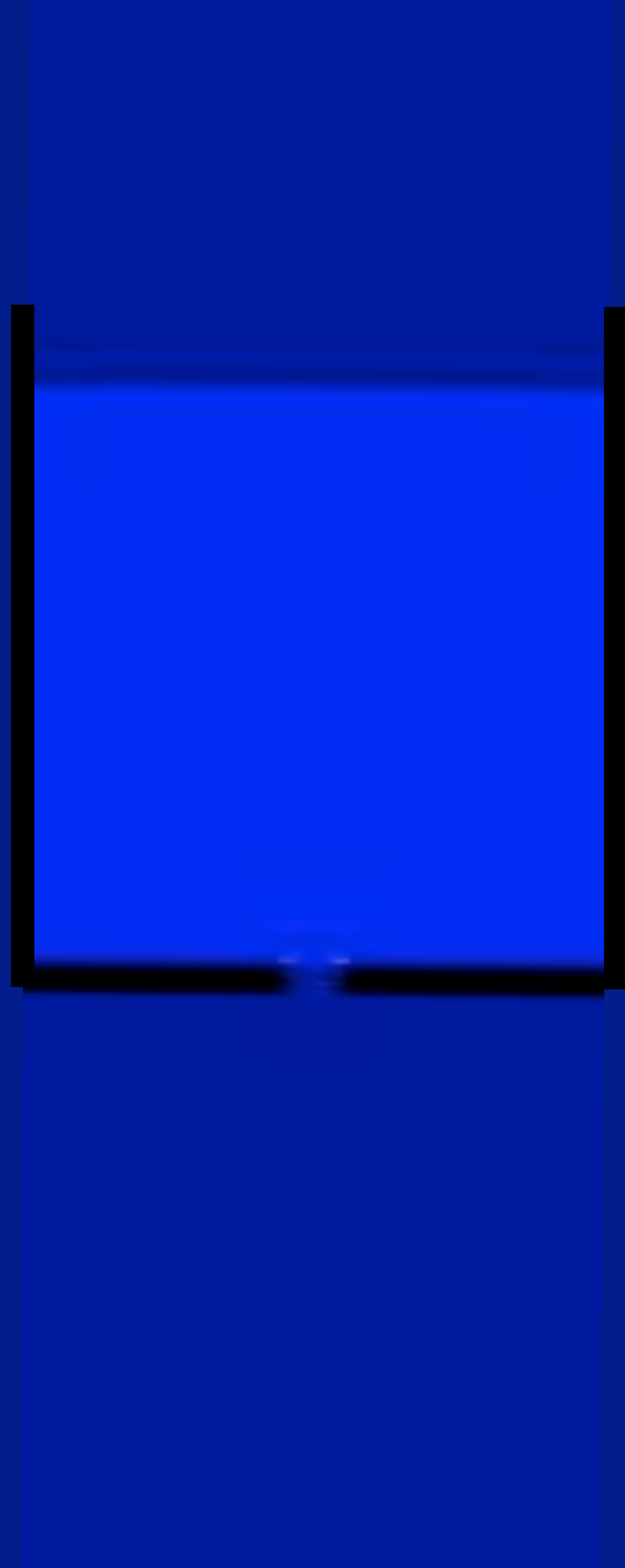
Yixian Zhou
Luke Fullard
Sylvain Viroulet
Francesco de Vita
Julien Philippi
Nitharshini Thiruvalluvar

& Stéphane Popinet

∂’Alembert par Félix LECOMTE. Avant 1786, Le Louvre Lens



Time for questions?