Axisymmetric viscous interfacial waves

Basilsik/Gerris Users' Meeting at Princeton University

Palas Kumar Farsoiya and Ratul Dasgupta

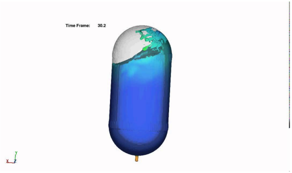
Department of Chemical Engineering
IIT Bombay

November 15, 2017

Motivation to study free oscillations capillary-gravity waves

(a) Droplet impact in deep pool (Source:https://www.saddlespace.org/williamsg/williamsscience/cms_page/view/28644289)

(b) Parasitic capillary waves on gravity waves (Source:http://homepage.ntu.edu.tw/~wttsai/surface_flows/surface_flows_in_nature.html)

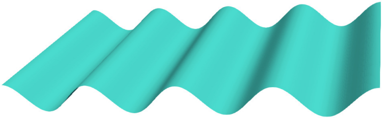


(c) Sloshing in tanks(Forced) (Source:https://www.youtube.com/watch?v=6xvDr20Tvvc)

(d) Faraday instability (Forced) (Source:https://www.flickr.com/photos/nonlin/18011742729)

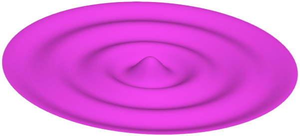
Related classical problems of capillary gravity waves

One Fluid- Effect of upper fluid neglected



(a) Planar wave

Two Fluids - Both fluids are considered

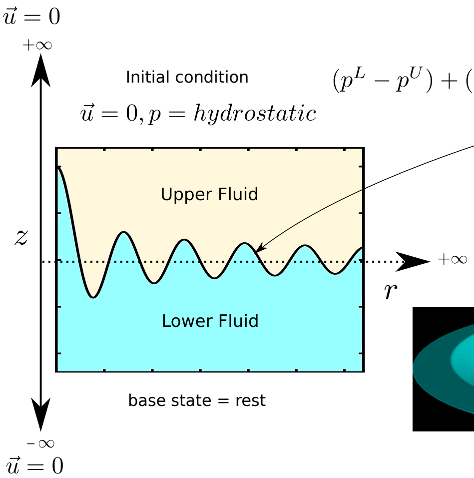


(b) Axisymmetric wave

1. One fluid inviscid and inviscid Lamb(1932) & Harrison(1908) - Normal mode
2. Two fluid Chandrasekhar(1961) - Normal mode
3. One fluid by Prosperetti(1976) - initial value problem
4. Two fluid by Prosperetti(1981) - initial value problem

1. Inviscid Cauchy-Poisson(1813) - initial value problem
2. Viscous one fluid by Miles(1968) - initial value problem
3. Present work - two fluids initial value problem

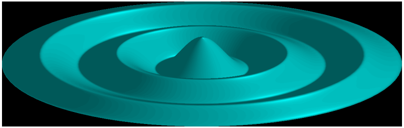
Two fluid axisymmetric Normal mode analysis and initial value problem



$$(p^L - p^U) + (n \cdot \tau \cdot n)^U - (n \cdot \tau \cdot n)^L = \sigma(\nabla \cdot n)$$

$$\vec{u}^U = \vec{u}^L$$

$$(t \cdot \tau \cdot n)^U = (t \cdot \tau \cdot n)^L$$



3D visualization

Palas Kumar Farsoiya, Y. S. Mayya and Ratul Dasgupta, Axisymmetric viscous interfacial oscillations - theory and simulations, *J. Fluid Mechanics,2017, 826, 797-818*

Solution Methodology

Solution

Linear approximation
$\epsilon \equiv a(0)k << 1$

Normal mode

$$b(r,z,t) = e^{\sigma t} f(z) J_\nu(kr)$$

Dispersion relation

$$\phi(\sigma, k) = 0$$

Initial Value

$$b(r,z,t) = f(z,t) J_\nu(kr)$$

Palas Kumar Farsoiya, Y. S. Mayya and Ratul Dasgupta, Axisymmetric viscous interfacial oscillations - theory and simulations, *J. Fluid Mechanics,2017, 826, 797-818*

Initial Value Problem-Solution(Potential)

Similar decomposition earlier by Lamb(1932), Miles(1968) and Prosperetti(1977)

Helmholtz Theorem

$$\vec{v} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{\psi}$$

$$\vec{v} = \vec{v}_p + \vec{v}_v$$

Potential part (curl free)

$$\vec{v}_p = \vec{\nabla}\phi$$

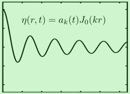
Continuity

$$\vec{\nabla} \cdot \vec{v}_p = 0$$

$$\vec{\nabla}^2 \phi = 0$$ Governing Equation

Boundary Conditions

$$\phi^U(r, \infty, t) \to 0$$
$$\phi^L(r, -\infty, t) \to 0$$
$$\left.\frac{\partial \phi^U}{\partial z}\right|_{z=0} = \left.\frac{\partial \phi^L}{\partial z}\right|_{z=0} = \frac{\partial \eta}{\partial t}$$



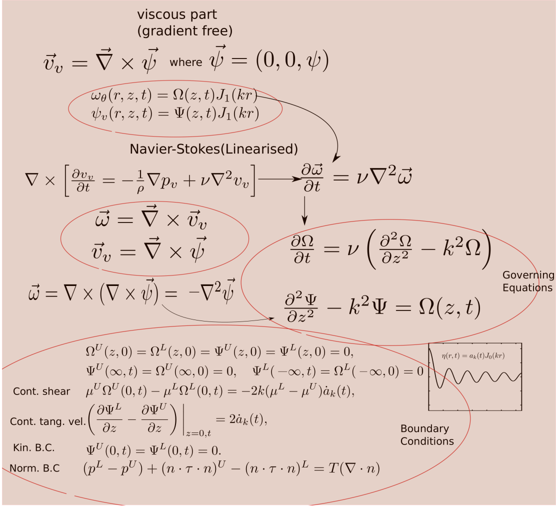
$$\phi^U(r, z, t) = -k^{-1} \exp[-kz] J_0(kr) \dot{a}_k(t),$$
$$\phi^L(r, z, t) = k^{-1} \exp[kz] J_0(kr) \dot{a}_k(t)$$

Solution

Palas Kumar Farsoiya, Y. S. Mayya and Ratul Dasgupta, Axisymmetric viscous interfacial oscillations - Theory and Simulations, *J. Fluid Mechanics,2017, 826, 797-818*

Initial Value Problem-Solution(Viscous)

Similar decomposition earlier by Lamb(1932), Miles(1968) and Prosperetti(1977)



Palas Kumar Farsoiya, Y. S. Mayya and Ratul Dasgupta, Axisymmetric viscous interfacial oscillations - Theory and Simulations, *J. Fluid Mechanics,2017, 826, 797-818*

Initial Value Solution in Laplace domain

Amplitude

$$\tilde{a}_k(s) = \frac{\left[s + \frac{1}{\rho^{\mathcal{U}}+\rho^{\mathcal{L}}}\left\{2k^2(\mu^{\mathcal{U}}+\mu^{\mathcal{L}}) + k(\mu^{\mathcal{U}}\zeta + \mu^{\mathcal{L}}\xi) - 2k^2\left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}}+k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}}+k}\right)\right\}\right]}{s^2 + \frac{1}{\rho^{\mathcal{U}}+\rho^{\mathcal{L}}}\left\{2k^2(\mu^{\mathcal{U}}+\mu^{\mathcal{L}}) + k(\mu^{\mathcal{U}}\zeta + \mu^{\mathcal{L}}\xi) - 2k^2\left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}}+k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}}+k}\right)\right\}s + \omega_0^2} a_k(0)$$

Similar expression derived for planar waves by Prosperetti A, *Physics of fluids,(1981)*

Vorticity

$$\tilde{\Omega}^{\mathcal{U}}(z,s) = \tilde{A}^{\mathcal{U}}(s)\exp\left[-z\lambda_{\mathcal{U}}\right], \quad \tilde{\Omega}^{\mathcal{L}}(z,s) = \tilde{A}^{\mathcal{L}}(s)\exp\left[z\lambda_{\mathcal{L}}\right]$$

Streamfunction

$$\tilde{\Psi}^{\mathcal{U}}(z,s) = \frac{\tilde{A}^{\mathcal{U}}(s)}{\lambda_{\mathcal{U}}^2 - k^2}\left(\exp[-z\lambda_{\mathcal{U}}] - \exp[-kz]\right)$$

$$\tilde{\Psi}^{\mathcal{L}}(z,s) = \frac{\tilde{A}^{\mathcal{L}}(s)}{\lambda_{\mathcal{L}}^2 - k^2}\left(\exp[z\lambda_{\mathcal{L}}] - \exp[kz]\right)$$

Pressure

$$\tilde{p}_v^{\mathcal{L}}(r,z,s) = -\mu^{\mathcal{L}}\tilde{A}^{\mathcal{L}}(s)J_0(kr)\exp[kz]$$

$$\tilde{p}_v^{\mathcal{U}}(r,z,s) = -\mu^{\mathcal{U}}\tilde{A}^{\mathcal{U}}(s)J_0(kr)\exp[-kz]$$

Initial Value solution in time domain ($\nu^{\mathcal{L}} = \nu^{\mathcal{U}}$)

Amplitude

$$\frac{a_k(t)}{a_k(0)} = \frac{4(\nu k^2)^2\,(1-4\beta)}{8(\nu k^2)^2\,(1-4\beta)+\omega_0^2}\mathrm{Erfc}\left(\sqrt{\nu k^2 t}\right) + \sum_{i=1}^{4}\frac{\hat{A}_i\hat{h}_i\omega_0^2\exp[(\hat{h}_i^2-\nu k^2)t]\mathrm{Erfc}\,(\hat{h}_i\sqrt{t})}{\nu k^2-\hat{h}_i^2}$$

where $\hat{h}_i \equiv h_i\sqrt{\omega_0}$ is the root of the equation

$$\hat{h}^4 - 4(\nu k^2)^{1/2}\beta\hat{h}^3 + 2(\nu k^2)(1-6\beta)\hat{h}^2 + 4(\nu k^2)^{3/2}(1-3\beta)\hat{h} + (\nu k^2)^2(1-4\beta) + \omega_0^2 = 0$$

Similar expression derived for planar waves by Prosperetti A, *Physics of fluids,(1981)*

Vorticity

$$\Omega^{\mathcal{U}}(z,t) = \left(\frac{\nu^{-1/2}}{\rho^{\mathcal{U}}+\rho^{\mathcal{L}}}\right)\left[\int_0^t dm\,\frac{\ddot{a}_k(m)\exp\left[-\left(\nu k^2(t-m)+\frac{z^2}{4\nu(t-m)}\right)\right]}{\sqrt{\pi(t-m)^3}}\left(zk\rho^{\mathcal{U}} + \frac{\rho^{\mathcal{L}}}{2}H_2\left(\frac{z}{2\sqrt{\nu(t-m)}}\right)\right)\right]$$

$$\Omega^{\mathcal{L}}(z,t) = \left(\frac{\nu^{-1/2}}{\rho^{\mathcal{U}}+\rho^{\mathcal{L}}}\right)\left[\int_0^t dm\,\frac{\ddot{a}_k(m)\exp\left[-\left(\nu k^2(t-m)+\frac{z^2}{4\nu(t-m)}\right)\right]}{\sqrt{\pi(t-m)^3}}\left(-zk\rho^{\mathcal{L}} + \frac{\rho^{\mathcal{U}}}{2}H_2\left(\frac{-z}{2\sqrt{\nu(t-m)}}\right)\right)\right]$$

Normal Mode solution

Dispersion Relation

$$\omega_0^2 + \sigma^2 = \frac{-\sigma}{(\rho^{\mathcal{L}} + \rho^{\mathcal{U}})} \left[2k^2 \left(\mu^{\mathcal{L}} + \mu^{\mathcal{U}} \right) + k \left(\mu^{\mathcal{U}} \xi + \mu^{\mathcal{L}} \xi \right) - 2k^2 \left(\frac{\mu^{\mathcal{U}} \xi}{\lambda^{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}} \zeta}{\lambda^{\mathcal{L}} + k} \right) \right]$$

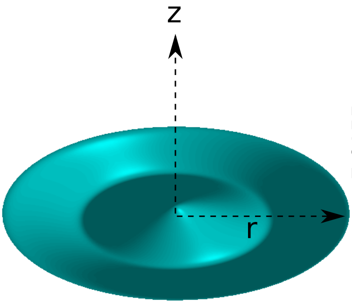
where,

$$\zeta(s) \equiv \frac{2(k + \lambda^{\mathcal{U}})(\mu^{\mathcal{U}} k + \mu^{\mathcal{L}} \lambda^{\mathcal{L}})}{\mu^{\mathcal{L}}(k + \lambda^{\mathcal{L}}) + \mu^{\mathcal{U}}(k + \lambda^{\mathcal{U}})}, \quad \xi(s) \equiv \frac{2(k + \lambda^{\mathcal{L}})(\mu^{\mathcal{L}} k + \mu^{\mathcal{U}} \lambda^{\mathcal{U}})}{\mu^{\mathcal{L}}(k + \lambda^{\mathcal{L}}) + \mu^{\mathcal{U}}(k + \lambda^{\mathcal{U}})}.$$

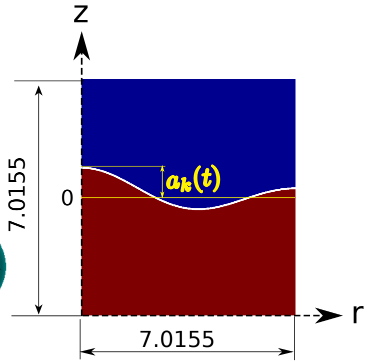
and $\lambda^{\mathcal{U}} \equiv \sqrt{k^2 + s/\nu^{\mathcal{U}}}$, $\lambda^{\mathcal{L}} \equiv \sqrt{k^2 + s/\nu^{\mathcal{L}}}$

Similar expression derived for planar waves by Chandrashekhar S, *Hydrodynamic and hydromagnetic stability,(1961)*

Simulation geometry



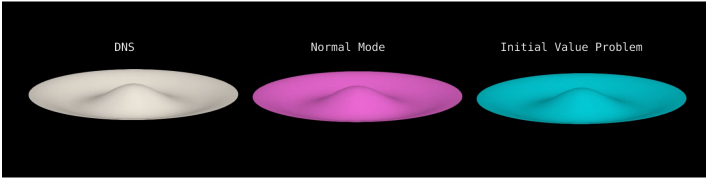
(a) Perturbed interface - 3D visualisation



(b) Perturbed interface - 2D

Direct Numerical Simulations (DNS)
using an open source solver Basilisk [*basilisk.fr*, Popinet S.]

Solution



Comparison of solutions

Cauchy-Poisson problem

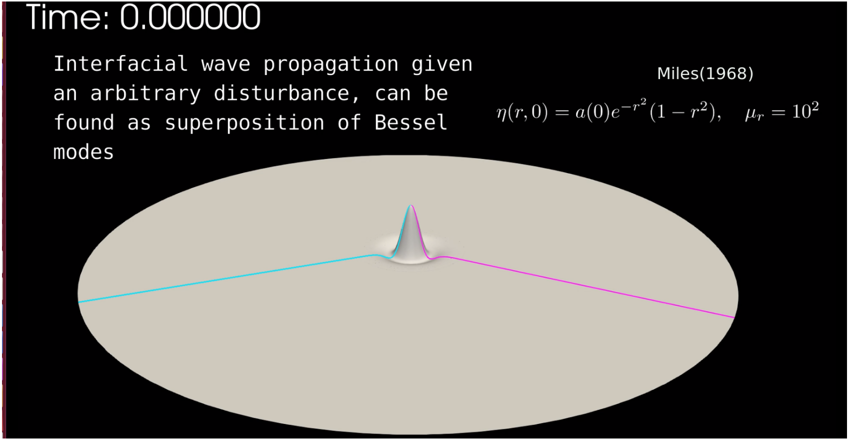
(a) Augustin-Louis Cauchy (1789-1857)

(b) Siméon Denis Poisson (1781-1840)

1. In December 1813, the French Académie des Sciences announced a mathematical prize competition on surface wave propagation on liquid of indefinite depth.
2. In July 1815, 25-year-old Augustin-Louis Cauchy submitted his entry.
3. Siméon D. Poisson, one of the judges, deposited a memoir of his own to record his independent work.
4. Cauchy was awarded the prize in 1816.
5. Poisson's memoir was published in 1818.

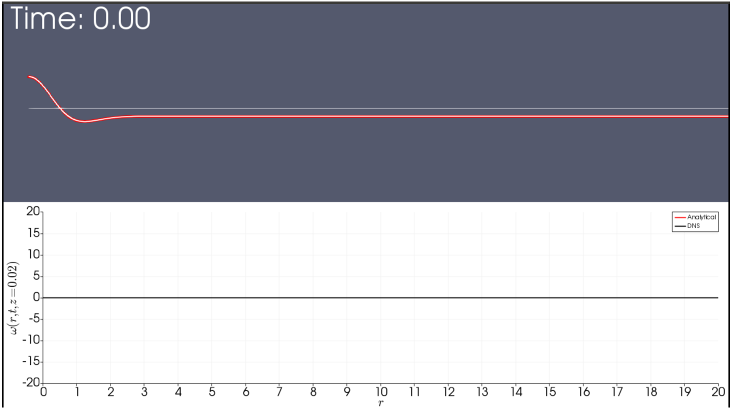
Source: Craik A. D. D, The origins of water wave theory, Annual reviews fluid mechanics (2004), 36:1-28

Cauchy-Poisson problem for viscous axisymmetric interface



Interfacial(two-fluid) Cauchy-Poisson problem(above)
Free surface(one fluid) Cauchy-Poisson problem, Miles J.W, [*J. Fluid Mech*,(1968)]

Vorticity

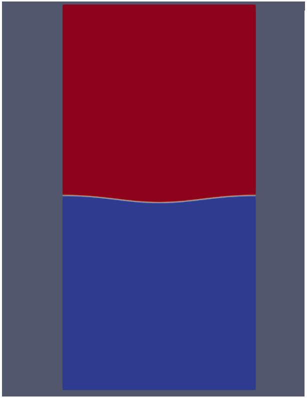


Time: 0.00
Analytical
DNS
20
15
10
5
0
-5
-10
-15
-20
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
r
$\omega(r,t,z=0.02)$

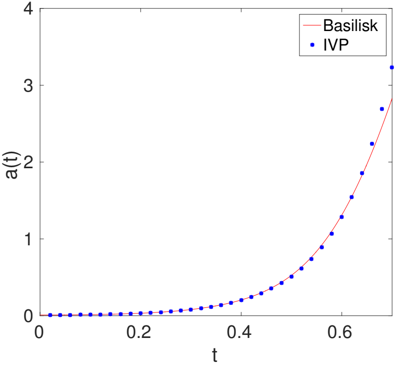
Non-linear behavior

Increasing $\epsilon \approx 1$

Rayleigh-Taylor instability with viscosity



(a) Rayleigh-Taylor (axisymmetric)



(b) amplitude vs time

$$\rho_u = 1,\ \rho_l = 0.01,\ \mu_u = 0.1,\ \mu_l = 0.001,\ T = 10,\ g = 100,$$
$$a(0) = 0.01$$

Conclusions

1. For low viscosity ratios the normal mode approximation agrees reasonably with DNS and IVP
2. For high viscosity ratios the initial value problem's solution predicts the intermediate transient behavior better than normal mode approximation
3. Non-linearity redistributes the energy among the various modes, consequently jetting and breakup is observed.

Acknowledgements

1. Prof. Y S Mayya (Chemical Engineering, IIT Bombay, India)
2. Industrial Research and Consultancy center (IRCC)
3. Dept. of Science and Technology, Govt. of India

Thank you